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Structure and properties of the solution space of general anisotropic laminates

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Abstract

The algebraic structure of the solution space of all types of anisotropic laminates is determined. The full space is shown to be the direct sum of a number of orthogonal eigenspaces, one for each simple or multiple eigenvalue, whose dimension equals the multiplicity. There are eight different types of eigenvalues, which combine to yield eleven distinct types of laminates with peculiar representations of the general solution. All such representations are explicitly obtained, along with the pseudo-metrics based on the binary product of the eigenvectors. This leads to the projection operators in the solution space, spectral sums and intrinsic tensors analogous to the Stroh–Barnett–Lothe tensors in 2-D elasticity. The present theoretical results are obtained by adopting a mixed formulation involving the deflection function and Airy's stress function, and by using new laminate elasticity matrices different from the conventional stiffness matrices **A**, **B** and **D**. The new formulation also discloses an isomorphism relating each anisotropic laminate to an image laminate, such that every equilibrium solution of the former directly yields an image solution of the latter by interchanging the kinematical and kinetic variables and the in-plane and out-of-plane variables. This implies, in particular, that the classical bending theory of homogeneous plates and symmetric laminates is not a distinct subject, despite its historical development and pedagogical recognition, but is mathematically identical to the plane stress problem of anisotropic elasticity.
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1. Introduction

Laminated plates, generally distinguished by strong anisotropy and coupling of the in-plane and out-of-plane responses, occupy a central place in advanced composite structures. Most textbooks on the mechanics of composite materials devote a significant portion of space to the theory and analysis of laminated plates. There is a voluminous literature on the subject, including extensive analytical and numerical solutions of various types of laminates subjected to a variety of loads and boundary conditions. However, theoretical

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studies on the mathematical properties and structure of the solutions of anisotropic laminates are scarce, when compared to the similar studies in two closely related areas: homogeneous isotropic plates and two-dimensional anisotropic elasticity. While the theory of anisotropic elasticity has received considerable attention since the pioneering work of Lekhnitskii (1963) and Stroh (1958), culminating in the recent publication of a major reference work by Ting (1996), no comparable development at the theoretical level has appeared on the subject of anisotropic laminates. More than half a century after its publication, Lekhnitskii's book (1968, first Russian edition 1944) remains the standard reference in the field.

Lekhnitskii obtained the general solutions of anisotropic laminates for the case when the material eigenvalues are all distinct. Coupling between in-plane and out-of-plane deformation was not considered. This results in a general solution containing two arbitrary analytic functions of two distinct complex variables and their complex conjugates. Many important problems of anisotropic plates, including elliptical plates and infinite plates with elliptical holes or elliptic inclusions, as well as rectangular plates and triangular plates, have been solved exactly or approximately using the complex variable representation and Lekhnitskii's method.

In the closely related problem of 2-D anisotropic elasticity, Lekhnitskii (1963) obtained the general solution of the stress potentials in terms of the anisotropic elastic compliances. In contrast, the formalism of Eshelby et al. (1953) and Stroh (1958, 1962) was based on the anisotropic elastic moduli and used the displacement functions as the primary unknown variables. Both formalisms were developed under the restrictive assumption that the eigenvalues are all distinct. In their study of coupled anisotropic plates, Lu (1994) and Lu and Mahrenholtz (1994) adopted the ERSS (Eshelby–Reed–Shockley–Stroh) formalism for the in-plane variables, and combined it with the out-of-plane deformation through bending–stretching coupling. The primary unknown variables are the three components of the mid-plane displacement, u, v and w, and the constitutive relations are represented by the three stiffness matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of the classical laminated plate theory. Their work extended Lekhnitskii's analysis of symmetric laminates to coupled laminates. However, it was also restricted by the assumption of distinct eigenvalues. The various degenerate cases with repeated eigenvalues were not investigated. The equations characterizing the eigenvalue problem were given in terms of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$, but the eigenvectors and eigensolutions were not obtained in explicit forms, due to the inherent complexity of the ERSS formalism.

The main objective of the present paper is twofold. First, Lekhnitskii's complex variable method for two-dimensional elasticity and the bending of uncoupled anisotropic plates will be extended to general anisotropic laminates with coupling between in-plane and out-of-plane deformation. This step is of considerable importance because laminate configurations that are asymmetric with respect to the mid-plane, and hence exhibiting the coupling effect, are found in newer applications of advanced composites, if not by design then occasionally due to degradation and damage such as delamination. We assume that both the in-plane and out-of-plane deformations are small so that, besides constitutive coupling, there is no coupling effect associated with geometrical nonlinearity. While the problem is basically identical to the one treated by Lu and Mahrenholtz, the present formulation adopts the Lekhnitskii formalism rather than the ERSS formalism in so far as it regards the in-plane deformation. This leads to a reduced eigenvalue problem of a lower dimension, and to simple, analytical expressions of the eigenvectors and eigensolutions that may be given explicitly. The second objective of this paper is to fully develop the complex variable method to include all degenerate cases, i.e., when the laminate has repeated eigenvalues and when the original Lekhnitskii method does not provide the full set of eigenvectors. In such cases, higher-order eigenvectors must be obtained to make up for the deficiency. The matter is far from merely academic because isotropic and transversely isotropic plates are degenerate. With regard to both objectives, the result of the present investigation is complete. No particular case is left unresolved.

In recent works on two-dimensional elasticity (Yin, 1997, 2000), it was shown that the formulation in terms of the elastic compliance coefficients, as initiated by Lekhnitskii, has decisive advantages over the displacement formulation using the anisotropic stiffness. In the first approach, the determination of eigensolutions is reduced to the eigenvalue problem of a 2×2 matrix $\mathbf{M}(\mu)$, which may be solved effortlessly to give simple analytical expressions of the eigenvectors. The ERSS formalism leads to a 3×3 characteristic matrix, and to lengthy analytical expressions even for the zeroth-order eigenvectors. Although he made no reference to the Russian work, Stroh gave dual derivations of the eigenvalue problem, one in terms of the anisotropic moduli and the other in terms of the elastic compliances, and commented on the advantage of the latter method in providing explicit expressions of the eigenvectors for the stress potentials. He even said to prefer to express the results in terms of such eigenvectors rather than the eigenvectors of displacements, and that the experimental data of anisotropic compliances were more readily available. Hence the term "Stroh formalism", which has recently been widely used to characterize a formalism with a strong bias toward the use of anisotropic moduli and displacement eigenvectors, notwithstanding the complexity of analysis and results, is historically less than accurate. It has been shown (Yin, 2000) that the algebraic complexity of the stiffness-based formulation becomes more acute in the various degenerate cases, where the higher-order eigenvectors must be obtained through relations that involve *all* lower-order eigenvectors that share the same eigenvalue.

For the present study of general anisotropic laminates, the best formulation is to use as the primary unknowns the in-plane stress function $F(x,y)$, i.e., Airy's function, and the deflection function $w(x,y)$. This choice appears obvious in view of the wide use of F in two-dimensional elasticity, and of w in the classical plate theory. Indeed, F and w are the preferred choice of the variables in von Karman's theory of plates, where the coupling between the in-plane and the out-of-plane deformation arises not from the constitutive relation but from geometrical nonlinearity. However, just as the ERSS formalism departs from the general use of F in 2-D elasticity, many existing works in classical laminate theory either use the three displacement functions u, v and w as the primary unknowns, or use the purely kinetic variables including the stress and moment resultants. This results in complicated equations, unavailability of general solutions except in restricted cases, and an obscure analytical formulation in which the fascinating algebraic structure and properties of the solution space remain largely unexplored and unrecognized.

If F and w and their second derivatives—the membrane forces and curvatures—are to be taken as the primary unknown variables, then the constitutive relations must express the complementary variables, the in-plane strains and the bending and twisting moments, in terms of the primary unknowns. The conventional stiffness matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of anisotropic laminates (Christensen, 1991), which express the kinetic variables of the moments and membrane forces in terms of the purely kinematical variables of the curvatures and membrane strains, are ill-suited for the analytical task of determining the eigensolutions. They will be replaced by new elasticity matrices $\mathbf{A}^*$, $\mathbf{B}^*$ and $\mathbf{D}^*$ expressing the moments and the in-plane strains in terms of the curvatures and the membrane forces. In this new formulation of anisotropic laminates, the determination of the eigensolutions is again reduced to a trivial eigenvalue problem of a 2×2 characteristic matrix $\mathbf{M}(\mu)$. Although the eigenvectors of the present problem have a higher dimension, many of the analytical results and expressions turn out to be formally identical to those of the 2-D anisotropic elasticity.

There are five distinct types of eigenvalues in 2-D anisotropic elasticity with the multiplicity varying from one to three. Their various combinations result in a classification of all anisotropic materials into five different types. For the general problem of coupled anisotropic laminates, the present analysis yields eight types of eigenvalues with the multiplicity varying from one to four, and eleven types of laminates. Each type of laminate has a distinct representation of the general solution. The types of eigenvalues depend on the multiplicity, and on whether the eigenvalue is *normal* ($\mathbf{M}(\mu) \neq \mathbf{0}$), *abnormal* ($\mathbf{M}(\mu) = \mathbf{0}$), or *super-abnormal* ($\mathbf{M}(\mu) = \mathbf{M}'(\mu) = \mathbf{0}$). If an eigenvalue possesses a smaller number of independent (zeroth-order) eigenvectors than its multiplicity, then higher-order eigenvectors must be found to make up for the deficiency in the representation of the general solution. These higher-order eigenvectors and the associated higher-order eigensolutions are derived in Section 2 of the present paper. It is found, a posteriori, that they can be obtained by differentiating appropriate analytical expressions of the zeroth-order eigenvectors and eigensolutions with respect to the parameter μ, followed by evaluation at the multiple eigenvalue. The presentation in Section 2 is very terse. A full exposition may be found in Yin (2003).

In Sections 3–5, the mathematical structure of the solution space of all eleven types of anisotropic laminates is investigated. It is shown first that, for every laminate, the eight-dimensional solution space is the direct sum of a number of orthogonal subspaces (to be called *eigenspaces*), one for each distinct eigenvalue, whose dimension equals the multiplicity of the eigenvalue. Orthogonality is defined with respect to the *binary product* of Eq. (26), which generally yields a complex number for a pair of eigenvectors sharing the same eigenvalue. Although the binary product is not an inner product, it endows the solution space and its orthogonal subspaces with a nonsingular *pseudo-metric*. This confirms the linear independence of the eigenvectors and eigensolutions. For a laminate with unequal eigenvalues, the eigenspaces are one-dimensional, and orthogonality of eigenvectors is as easily proved as in the corresponding case of 2-D anisotropic elasticity. Proof of the orthogonality of eigenspaces is not trivial for degenerate, extra-degenerate and ultra-degenerate laminates. Furthermore, the inner structure of the eigenspace associated with a multiple eigenvalue, as characterized by the pseudo-metric, is significantly different between a *normal* eigenvalue and an *abnormal* eigenvalue. The pseudo-metrics of the eigenspaces associated with eight different types of eigenvalues are obtained in Section 4 and listed in Appendix A. They form the basis of the discrete spectral analysis in Section 5, and of the representations of general solutions and intrinsic tensors for all eleven types of laminates as given in Section 6.

When the new elasticity matrices $\mathbf{A}^*$, $\mathbf{B}^*$ and $\mathbf{D}^*$ are used instead of the conventional stiffness matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$, and when appropriate changes are made in the *order* and in certain algebraic signs of the components of the moment and the curvature, the coupled differential equations governing the redefined variables reveal full symmetry with regard to two groups of variables: one group consisting of $-u$, $-v$ and F, and the other group comprising the moment potentials and the deflection function w. In this symmetry relationship, the kinematical variables of membrane strains change into the kinetic variables of bending and twisting moments, and the membrane forces change into the curvature components. This isomorphism associates each anisotropic laminate that has the elasticity matrices $\mathbf{A}^*$, $\mathbf{B}^*$ and $\mathbf{D}^*$ with an *image* laminate having the corresponding elasticity matrices $\mathbf{D}^*$, $-\mathbf{B}^{*\mathrm{T}}$ and $-\mathbf{A}^*$, such that every equilibrium solution of the original laminate is transformed into a corresponding equilibrium solution of the image laminate by the dual interchange of the kinematical variables with the kinetic variables and the in-plane variables with the out-of-plane variables. In particular, kinematical boundary conditions of the in-plane displacements are mapped into kinetic boundary conditions of the moment potentials, and vice versa.

This isomorphism between the image laminates emerges clearly in the present formulation which uses the new elasticity matrices $\mathbf{A}^*$, $\mathbf{B}^*$ and $\mathbf{D}^*$. It is obscured in the conventional theory of laminates that uses the stiffness matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$. For a laminate with no bending/stretching coupling, the symmetry relation implies that all in-plane elasticity solutions may be converted into the bending solutions of the image laminate. From both theoretical and computational points of view, the plane-stress elasticity problem, whether isotropic or anisotropic, is essentially identical to the bending problem of classical thin plates, and the latter should never have been developed, and continue to be taught, as if it were a distinct subject. In other words, the totality of bending solutions of anisotropic laminates with mid-plane symmetry is coextensive with the totality of 2-D plane-stress anisotropic elasticity solutions. Hence the rich inventory of isotropic and anisotropic plane-stress solutions, including those contained in the works of Muskhelishvili (1963) and Lekhnitskii (1963), are easily converted to corresponding bending/twisting solutions of the image laminates, but the two sets of solutions have boundary conditions of the complementary types.

The results and proofs in this paper are established in a general mathematical form using combinatorials with variable indices, so that their validity depends neither on the dimensions of vectors and matrices nor on the multiplicity of roots. Hence the present proofs and results (particularly Sections 3–5 on the orthogonality of eigenspaces, pseudo-metrics and projection operators, which are inadequately treated in Yin (2000)) may apply to 2-D anisotropic elasticity by merely changing the dimension of eigenvectors from eight to six. They may also be applicable to other related problems of anisotropic media having a similar mathematical structure, such as problems of surface waves and piezoelectric materials. The principal requirement is that the primary unknown variables—curvatures and membrane forces in the present case—are derivable as the components of the second gradients of scalar functions (w and F). This important relation, however, remains unexploited in the ERSS formalism, which accounts for its unwieldiness.

2. Eigensolutions

In the conventional theory of anisotropic laminated plates, the constitutive relations are given in terms of three symmetric, 3×3 stiffness matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ (Christensen, 1991):

$$\mathbf{n} = \mathbf{A}\boldsymbol{\epsilon} + \mathbf{B}\boldsymbol{\kappa}, \quad \mathbf{m} = \mathbf{B}\boldsymbol{\epsilon} + \mathbf{D}\boldsymbol{\kappa}, \tag{1}$$

where

$$\boldsymbol{\epsilon} = \{\epsilon_x, \epsilon_y, 2\epsilon_{xy}\}^{\mathrm{T}} = \{u_{,x}, v_{,y}, u_{,y} + v_{,x}\}^{\mathrm{T}}, \quad \boldsymbol{\kappa} = \{w_{,xx}, w_{,yy}, 2w_{,xy}\}^{\mathrm{T}}, \tag{2a, b}$$

$$\mathbf{n} = \{N_x, N_y, N_{xy}\}^{\mathrm{T}}, \quad \mathbf{m} = \{M_x, M_y, M_{xy}\}^{\mathrm{T}}, \tag{2c, d}$$

and u and v are the mid-plane tangential displacements and w is the deflection function. The equilibrium equations imply that the stress and moment resultants be derivable from three potential functions $F(x, y)$, $\Psi_1(x, y)$ and $\Psi_2(x, y)$:

$$N_x = F_{,yy}, \quad N_y = F_{,xx}, \quad N_{xy} = -F_{,xy}, \tag{3}$$

$$M_y = \Psi_{1,x}, \quad M_x = \Psi_{2,y}, \quad -2M_{xy} = \Psi_{1,y} + \Psi_{2,x}. \tag{4}$$

Let

$$\boldsymbol{\chi} \equiv \{w_{,y}, -w_{,x}, F_{,y}, -F_{,x}, \Psi_1, \Psi_2, -u, -v\}^{\mathrm{T}}, \tag{5}$$

$$\boldsymbol{\phi} \equiv \{w_{,yy}, w_{,xx}, -w_{,xy}, F_{,yy}, F_{,xx}, -F_{,xy}\}^{\mathrm{T}}, \tag{6a}$$

$$\boldsymbol{\theta} \equiv \{M_y, M_x, -2M_{xy}, -\epsilon_x, -\epsilon_y, -2\epsilon_{xy}\}^{\mathrm{T}}. \tag{6b}$$

Then Eq. (1) may be rewritten as

$$\boldsymbol{\theta} = \mathbf{C}^*\boldsymbol{\phi}, \tag{7}$$

where

$$\mathbf{C}^* \equiv \begin{bmatrix} \mathbf{D}^* & \mathbf{B}^* \\ \mathbf{B}^{*\mathrm{T}} & -\mathbf{A}^* \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\Lambda}(\mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B})\boldsymbol{\Lambda} & \boldsymbol{\Lambda}\mathbf{B}\mathbf{A}^{-1} \\ \mathbf{A}^{-1}\mathbf{B}\boldsymbol{\Lambda} & -\mathbf{A}^{-1} \end{bmatrix}, \tag{8a}$$

$$\boldsymbol{\Lambda} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{8b}$$

Notice that $\mathbf{A}^*$ is the in-plane compliance matrix and the elements of $\mathbf{B}^*$ have the dimension of thickness. The following matrix functions are important to the present theory:

$$\mathbf{\Phi}(\mu) \equiv \begin{bmatrix} \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{E}(\mu) \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & 1 \end{bmatrix}, \tag{9a, b}$$

$$\mathbf{Y}(\mu) \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\mu & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\mu & 0 & 1 \end{bmatrix}, \quad \mathbf{P}(\mu) \equiv \begin{bmatrix} -\mu^2 & 0 \\ -1 & 0 \\ \mu & 0 \\ 0 & -\mu^2 \\ 0 & -1 \\ 0 & \mu \end{bmatrix}, \tag{10a, b}$$

$$\mathbf{J}_1(\mu) \equiv \begin{bmatrix} -\mu & 0 \\ 1 & 0 \\ 0 & -\mu \\ 0 & 1 \end{bmatrix}, \quad \mathbf{J}_2(\mu) \equiv \mathbf{Y}(\mu)\mathbf{C}^*\mathbf{P}(\mu), \quad \mathbf{J}(\mu) \equiv \begin{bmatrix} \mathbf{J}_1(\mu) \\ \mathbf{J}_2(\mu) \end{bmatrix}, \tag{11a, b, c}$$

$$\mathbf{M}(\mu) \equiv \mathbf{P}(\mu)^{\mathrm{T}}\mathbf{C}^*\mathbf{P}(\mu). \tag{12a}$$

The components of the matrix $\mathbf{M}(\mu)$ are quadratic functions of μ, i.e.,

$$\begin{aligned} M_{11}(\mu) &= \{-\mu^2, -1, \mu\}\mathbf{D}^*\{-\mu^2, -1, \mu\}^{\mathrm{T}}, \\ M_{12}(\mu) &= M_{21}(\mu) = \{-\mu^2, -1, \mu\}\mathbf{B}^*\{-\mu^2, -1, \mu\}^{\mathrm{T}}, \\ M_{22}(\mu) &= \{-\mu^2, -1, \mu\}(-\mathbf{A}^*)\{-\mu^2, -1, \mu\}^{\mathrm{T}}. \end{aligned} \tag{12b}$$

Consider first the zeroth-order eigensolutions, which have the following form

$$\boldsymbol{\chi} = f(x + \mu y)\boldsymbol{\xi}, \tag{13}$$

where $\boldsymbol{\xi}$ is a complex constant vector and f is an arbitrary analytic function involving a complex parameter μ. Substituting Eq. (13) into (5), and using $w_{,xy} = w_{,yx}$ and $F_{,xy} = F_{,yx}$, one obtains

$$(-1/\mu)\begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} = \begin{Bmatrix} \xi_3 \\ \xi_4 \end{Bmatrix} = \boldsymbol{\eta}. \tag{14}$$

Then Eqs. (6a,b) and (7) become

$$\boldsymbol{\phi} = f'(x + \mu y)\mathbf{\Phi}(\mu)\boldsymbol{\xi} = f'(x + \mu y)\mathbf{P}(\mu)\boldsymbol{\eta}, \quad \boldsymbol{\theta} = f'(x + \mu y)\mathbf{E}(\mu)\boldsymbol{\xi}, \tag{15a, b}$$

$$\mathbf{E}(\mu)\boldsymbol{\xi} = \mathbf{C}^*\mathbf{P}(\mu)\boldsymbol{\eta}. \tag{16}$$

Premultiplying Eq. (16) by the matrices $\mathbf{Y}$ and $\mathbf{P}^{\mathrm{T}}$ yields, respectively,

$$[\mathbf{0}_{4\times4}, \mathbf{I}_4]\boldsymbol{\xi} = \{\xi_5, \xi_6, \xi_7, \xi_8\}^{\mathrm{T}} = \mathbf{Y}(\mu)\mathbf{C}^*\mathbf{P}(\mu)\boldsymbol{\eta}, \tag{17}$$

$$\mathbf{M}(\mu)\boldsymbol{\eta} = \mathbf{0}, \tag{18}$$

where $\mathbf{0}_{4\times4}$ and $\mathbf{I}_4$ denote four-dimensional zero matrix and identity matrix, respectively. Combination of (14) and (17) yields an expression of the eight-dimensional vector $\boldsymbol{\xi}$ in terms of $\boldsymbol{\eta}$:

$$\boldsymbol{\xi} = \mathbf{J}(\mu)\boldsymbol{\eta}, \tag{19}$$

where the matrix function $\mathbf{J}(\mu)$ is defined by Eq. (11).

Eq. (18) has a nontrivial solution if μ is a root of the characteristic equation

$$\delta(\mu) \equiv \mathrm{Det}[\mathbf{M}(\mu)] = 0. \tag{20}$$

For each root of Eq. (20), Eq. (18) yields at least one nontrivial $\boldsymbol{\eta}$. Then Eqs. (19) and (13) give a nontrivial vector $\boldsymbol{\xi}$ and a solution $\boldsymbol{\chi}$. The roots of Eq. (20) are called eigenvalues, and $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ is, respectively, the associated eigenvector and eigensolution.

According to (12), Eq. (20) is a polynomial equation in μ of the eighth degree with real coefficients. Hence its complex roots occur in conjugate pairs. It has been shown (Yin, 2003) that the equation cannot have real roots if the elastic strain energy of the laminate is positive definite. If all four complex conjugate pairs of eigenvalues are distinct, then each eigenvalue yields an eigenvector and an eigensolution. An appropriate linear combination of the four complex conjugate pairs of eigensolutions gives the general solution with real values for the various physical quantities.

If Eq. (20) has multiple roots, then the preceding procedure may yield fewer than eight independent eigensolutions, and additional (higher-order) eigensolutions must be obtained to supplement the preceding (zeroth-order) eigensolutions. The form of these higher-order eigensolutions depends on the type and multiplicity of the eigenvalue. An eigenvalue μ is called *normal* if $\mathbf{M}(\mu)$ of Eq. (12) is not the null matrix. It is called *abnormal* if $\mathbf{M}(\mu)$ is the null matrix but $\mathbf{M}'(\mu)$ is not, and *superabnormal* if $\mathbf{M}(\mu) = \mathbf{M}'(\mu) = \mathbf{0}$. The multiplicity may vary from one to four for a normal eigenvalue, and from two to four for an abnormal eigenvalue. A superabnormal eigenvalue must be a quadruple root, because it is a root of all three scalar equations $M_{ij}(\mu) = 0$. Thus there are eight different types of eigenvalues, one of which is superabnormal, three are abnormal, and all others normal.

Consider an eigenvalue μ of multiplicity p ($2 \leqslant p \leqslant 4$). A Nth order eigensolution ($N < p$) has the following expression

$$\boldsymbol{\chi}^{[N]} = \sum_{0 \leqslant j \leqslant N} (N, j) y^j f^{(j)}(x + \mu y) \boldsymbol{\xi}^{[N-j]}, \tag{21}$$

where $(N, j) \equiv N!/\{(N-j)!j!\}$, $f^{(j)}$ denotes the jth derivative of an arbitrary analytic function f, and $\boldsymbol{\xi}^{[0]}, \boldsymbol{\xi}^{[1]}, \ldots, \boldsymbol{\xi}^{[N]}$ are eight-dimensional complex constant vectors. These constant vectors may be expressed in terms of two-dimensional vectors $\boldsymbol{\eta}^{[0]}, \boldsymbol{\eta}^{[1]}, \ldots, \boldsymbol{\eta}^{[N]}$ and the matrix $\mathbf{J}(\mu)$ in the following manner

$$\boldsymbol{\xi}^{[j]} = \mathbf{J}(\mu)\boldsymbol{\eta}^{[j]} + (j, 1)\mathbf{J}'(\mu)\boldsymbol{\eta}^{[j-1]} + (j, 2)\mathbf{J}''(\mu)\boldsymbol{\eta}^{[j-2]} + (j, 3)\mathbf{J}'''(\mu)\boldsymbol{\eta}^{[j-3]}, \tag{22}$$

where

$$\boldsymbol{\eta}^{[j]} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{\xi}^{[j]} \quad (0 \leqslant j \leqslant N)$$

and it is understood that $\boldsymbol{\eta}^{[j]} = \mathbf{0}$ if j is a negative integer. $\boldsymbol{\eta}^{[0]}, \boldsymbol{\eta}^{[1]}, \ldots,$ and $\boldsymbol{\eta}^{[N]}$ are determined by the following system of equations

$$\sum_{0 \leqslant j \leqslant p} (p, j)\mathbf{M}^{(j)}(\mu)\boldsymbol{\eta}^{[p-j]} = \mathbf{0} \quad (p = 0, 1, \ldots, N). \tag{23}$$

Here $\mathbf{M}^{(j)}$ denotes the jth derivative of the matrix defined by Eq. (12).

The identity involving $\mathbf{M}(\mu)$ and its adjoint matrix $\mathbf{W}(\mu)$

$$\mathbf{M}(\mu)\mathbf{W}(\mu) = \delta(\mu)\mathbf{I} \tag{24}$$

may be differentiated with respect to μ repeatedly to yield additional identities

$$\sum_{0 \leqslant j \leqslant N} (N, j)\mathbf{M}^{(N-j)}(\mu)\mathbf{W}^{(j)}(\mu) = \delta^{(N)}(\mu)\mathbf{I}. \tag{25}$$

Since $\delta^{(N)}(\mu) = 0$ whenever $0 \leqslant N \leqslant p-1$, one has

$$\sum_{0 \leqslant j \leqslant N} (N,j)\mathbf{M}^{(N-j)}(\mu)\mathbf{W}^{(j)}(\mu) = \mathbf{0} \quad (0 \leqslant N \leqslant p-1). \tag{26}$$

If μ_0 is a *normal* eigenvalue, so that the 2×2 symmetric matrix $\mathbf{W}(\mu_0)$ is of rank one, then $\mathbf{W}(\mu_0)$ has at least one nonvanishing diagonal element. For otherwise $W_{11} = W_{22} = 0$ and then it would follow from $\mathrm{Det}[\mathbf{W}] = 0$ that $W_{12} = 0$, so that $\mathbf{W}(\mu_0)$ would be the zero matrix. We define the column selector $\boldsymbol{\rho}$ based on the position of the larger diagonal element of $\mathbf{W}(\mu_0)$:

$$\boldsymbol{\rho} = \begin{cases} \{1,0\}^{\mathrm{T}} & \text{if } |W_{11}(\mu_0)| \geqslant |W_{22}(\mu_0)|, \\ \{0,1\}^{\mathrm{T}} & \text{otherwise,} \end{cases} \tag{27a}$$

$$W \equiv \boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}(\mu_0)\boldsymbol{\rho}, \ldots, W''' \equiv \boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}'''(\mu_0)\boldsymbol{\rho}. \tag{27b}$$

The definition of $\boldsymbol{\rho}$ ensures that, for a normal eigenvalue μ_0, $\boldsymbol{\eta}(\mu_0) = \mathbf{W}(\mu_0)\boldsymbol{\rho}$ is always a nontrivial vector and it yields a zeroth-order eigenvector $\boldsymbol{\xi}^{[0]} = \mathbf{J}(\mu_0)\mathbf{W}(\mu_0)\boldsymbol{\rho}$. For a multiple normal eigenvalue, the system of equations (23) has the following solutions $(N \leqslant p \leqslant 1)$

$$\boldsymbol{\eta}^{[j]} = \mathbf{W}^{(j)}(\mu)\boldsymbol{\rho} \quad (j = 0, 1, \ldots, N), \tag{28}$$

where each $\mathbf{W}^{(j)}(\mu)$ is the adjoint matrix of $\mathbf{M}^{(j)}(\mu)$. Substituting (28) into (22), one obtains the higher-order eigenvectors $\boldsymbol{\xi}^{[j]}$. Then Eq. (21) gives the eigensolutions of the corresponding order containing an arbitrary analytic function $f(x + \mu y)$.

For an abnormal eigenvalue μ, $\mathbf{M}(\mu)$ vanishes so that Eq. (18) is trivially satisfied by an arbitrary $\boldsymbol{\eta}$. Two zeroth-order eigensolutions are obtained by choosing $\boldsymbol{\eta}$ to be $\{1,0\}^{\mathrm{T}}$ and $\{0,1\}^{\mathrm{T}}$ successively. The corresponding eigenvectors are the two columns of $\mathbf{J}(\mu_0)$, and Eq. (21) gives two zeroth-order eigensolutions.

If μ_0 is an *abnormal* eigenvalue of multiplicity three or four, then $\mathbf{M}(\mu_0) = \mathbf{W}(\mu_0) = \mathbf{0}$ and $\delta(\mu_0) = \delta'(\mu_0) = \delta''(\mu_0) = 0$, but $\mathbf{M}'(\mu_0)$ and $\mathbf{W}'(\mu_0)$ are not zero matrices. The relation $\delta''(\mu_0) = 0$ reduces to $2W'_{11}(\mu_0)W'_{22}(\mu_0) - 2W'_{12}(\mu_0)^2 = 0$. Hence $W'_{11}(\mu_0)$ and $W'_{22}(\mu_0)$ cannot both vanish; otherwise $\mathbf{W}'(\mu_0)$ would be the zero matrix. We define, for an abnormal eigenvalue μ_0,

$$\hat{\boldsymbol{\rho}} = \begin{cases} \{1,0\}^{\mathrm{T}} & \text{if } |W'_{11}(\mu_0)| \geqslant |W'_{22}(\mu_0)|, \\ \{0,1\}^{\mathrm{T}} & \text{otherwise,} \end{cases} \tag{29a}$$

$$W \equiv \hat{\boldsymbol{\rho}}^{\mathrm{T}}\mathbf{W}(\mu_0)\hat{\boldsymbol{\rho}} = 0, \quad W' \equiv \hat{\boldsymbol{\rho}}^{\mathrm{T}}\mathbf{W}'(\mu_0)\hat{\boldsymbol{\rho}}, \ldots, \quad W''' \equiv \hat{\boldsymbol{\rho}}^{\mathrm{T}}\mathbf{W}'''(\mu_0)\hat{\boldsymbol{\rho}}. \tag{29b}$$

Then $\boldsymbol{\eta}^{[1]} \equiv \mathbf{W}'(\mu_0)\hat{\boldsymbol{\rho}}$ is always a nontrivial vector, and it satisfies Eq. (23) for $N = 1$ in view of Eq. (26), which for $N = 1$ reduces to $\mathbf{M}'\mathbf{W} + \mathbf{M}\mathbf{W}' = \mathbf{0}$ since δ' vanishes for a repeated eigenvalue. The complete set of solutions of Eq. (23) is given by $(N \leqslant p-1)$

$$\boldsymbol{\eta}^{[j]} = \mathbf{W}^{(j)}(\mu)\hat{\boldsymbol{\rho}} \quad (1 \leqslant j \leqslant N). \tag{30}$$

The higher-order eigenvectors and eigensolutions are obtained by substituting (30) into (22) and then into (21).

A superabnormal eigenvalue μ_0 has the two zeroth-order eigenvectors given by the two columns of $\mathbf{J}(\mu_0)$. In addition, it has two first-order eigenvectors given by the two columns of $\mathbf{J}'(\mu_0)$.

For the eigensolutions of the various orders expressed by Eqs. (21) and (22), the curvatures, mid-plane strains and stress and moment resultants are given by

$$\boldsymbol{\phi}^{[N]} = y^N f^{(N+1)}\boldsymbol{\Phi}\boldsymbol{\xi}^{[0]} + \sum_{0 \leqslant j \leqslant N-1} (N,j) y^j f^{(j+1)}\{\boldsymbol{\Phi}\boldsymbol{\xi}^{[N-j]} + (N-j)\boldsymbol{\Phi}'\boldsymbol{\xi}^{[N-j-1]}\}, \tag{31a}$$

$$\boldsymbol{\theta}^{[N]} = y^N f^{(N+1)}\mathbf{E}\boldsymbol{\xi}^{[0]} + \sum_{0\leqslant j\leqslant N-1}(N,j)y^j f^{(j+1)}\{\mathbf{E}\boldsymbol{\xi}^{[N-j]} + (N-j)\mathbf{E}'\boldsymbol{\xi}^{[N-j-1]}\}. \tag{31b}$$

In the literature on anisotropic elasticity, the higher-order eigensolutions are often called generalized eigenfunctions. The term "eigensolution" is preferred because the real parts of all eigensolutions are indeed equilibrium solutions of the corresponding anisotropic laminated plate.

The zeroth-order eigenvector $\boldsymbol{\xi}^{[0]}$, obtained from Eqs. (18) and (19), is a polynomial function of μ to be evaluated at the specific eigenvalue. If one withholds the evaluation, treats μ instead as a variable, and differentiates Eqs. (13), (19) and (15a,b) repeatedly with respect to μ, followed by evaluation at the specific eigenvalue, one obtains precisely the same equations as (21), (22) and (31a,b) except that all jth-order quantities in the equations are replaced by the jth derivatives of the corresponding zeroth-order quantities with respect to μ. Therefore, the higher-order eigenvectors and eigensolutions can be obtained formally by repeated differentiation of appropriate analytical expressions of the zeroth-order eigenvectors and eigensolutions, followed by evaluation at the specific eigenvalue. This derivative rule (Yin, 2000, 2003) will be used frequently in the following analysis.

3. Orthogonal eigenspaces

The four complex conjugate pairs of eigenvalues will be arranged as a sequence $\boldsymbol{\mu} = \{\mu_1, \mu_2, \mu_3, \mu_4, \bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4\}$, such that the first four elements have positive imaginary parts and, among these four, any double or triple eigenvalue appears after all simple eigenvalues. A repeated eigenvalue appears in $\boldsymbol{\mu}$ as many times as its multiplicity. The eigenvectors will be assembled as the columns of a matrix in a one-to-one correspondence to the eigenvalues in $\boldsymbol{\mu}$ and, for those associated with a common multiple eigenvalue, arranged in the increasing order j. This yields an 8×8 *eigenmatrix* $\mathbf{Z}$, such that the last four columns of $\mathbf{Z}$ are the complex conjugates of the first four, i.e.,

$$\mathbf{Z} = \{\mathbf{Z}^+, \overline{\mathbf{Z}}^+\}. \tag{32}$$

For each anisotropic laminate, the eigenmatrix $\mathbf{Z}$ completely determines the general solution. It also determines certain real-valued matrices (analogous to the Stroh–Barnett–Lothe tensors in 2-D anisotropic elasticity) that characterize the intrinsic structure of the solution space.

The eight-dimensional solution space may be decomposed into a number of orthogonal subspaces, one for each distinct simple or multiple eigenvalue. These subspaces will be called *eigenspaces*. Orthogonality is defined with respect to the *binary product*, which may be introduced as follows for any two matrices $\mathbf{U}$ and $\mathbf{V}$ of row dimension eight (their column dimensions need not be equal):

$$[\![\mathbf{U},\mathbf{V}]\!] \equiv \mathbf{U}^{\mathrm{T}}\mathbf{II}\mathbf{V} = [\![\mathbf{V},\mathbf{U}]\!]^{\mathrm{T}}, \quad \text{where } \mathbf{II} \equiv \begin{bmatrix} \mathbf{0}_{4\times4} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{0}_{4\times4} \end{bmatrix}. \tag{33}$$

The binary product of a matrix $\mathbf{U}$ with itself yields the matrix $\mathbf{U}^{\mathrm{T}}\mathbf{II}\mathbf{U}$ which is always symmetric.

If $\mathbf{U}$ and $\mathbf{V}$ are submatrices of the eigenmatrix $\mathbf{Z}$, each consisting of a number of eigenvectors, then, under a rotation of the coordinates in the x–y plane, one has

$$\begin{Bmatrix} x^* \\ y^* \end{Bmatrix} = \mathbf{Q}_2\begin{Bmatrix} x \\ y \end{Bmatrix}, \quad \text{where } \mathbf{Q}_2 \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{34}$$

The eigenvectors and the submatrix $\mathbf{Z}^+$ transform in the following manner

$$\boldsymbol{\xi}^* = \mathbf{Q}_8\boldsymbol{\xi}, \quad (\mathbf{Z}^+)^* = \mathbf{Q}_8\mathbf{Z}^+, \tag{35}$$

and so do $\mathbf{U}$ and $\mathbf{V}$, where

$$\mathbf{Q}_8 \equiv \langle \mathbf{Q}_2, \mathbf{Q}_2, \mathbf{Q}_2, \mathbf{Q}_2\rangle \tag{36}$$

denotes the block diagonal matrix formed by four identical submatrices $\mathbf{Q}_2$. It is easily verified that

$$\mathbf{II}\;\mathbf{II} = \mathbf{I}_8, \quad \mathbf{Q}_8^{\mathrm{T}}\mathbf{II}\mathbf{Q}_8 = \mathbf{II}. \tag{37a, b}$$

Applying the binary product to two distinct eigenvectors ξ_1 and ξ_2, to the matrices $\mathbf{U}$ and $\mathbf{V}$, and to their images after rotation, one obtains, by using Eq. (37b)

$$[\![\xi_1^*, \xi_2^*]\!] = [\![\xi_1, \xi_2]\!], \quad [\![\mathbf{U}^*, \mathbf{V}^*]\!] = [\![\mathbf{U}, \mathbf{V}]\!]. \tag{38a, b}$$

Furthermore,

$$\mathbf{\Omega} \equiv [\![\mathbf{Z}^+, \mathbf{Z}^+]\!] = [\![(\mathbf{Z}^+)^*, (\mathbf{Z}^+)^*]\!]. \tag{39}$$

Therefore, the binary products of eigenvectors, and of the submatrices of the eigenmatrix $\mathbf{Z}$, are invariant under a rotation of the coordinates. Hence the matrix $\mathbf{\Omega}$ characterizes the intrinsic structure of the four-dimensional space spanned by the eigenvectors of $\mathbf{Z}^+$, and will be referred to as a *pseudo-metric* of that space. It possesses some properties of a metric, but it is complex-valued and certainly not positive definite.

For two eigensolutions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}'$, the binary product

$$[\![\boldsymbol{\chi}, \boldsymbol{\chi}']\!] = \begin{vmatrix} \Psi_1' & \Psi_2' \\ w_{,x} & w_{,y} \end{vmatrix} + \begin{vmatrix} -u' & -v' \\ F_{,x} & F_{,y} \end{vmatrix} + \begin{vmatrix} \Psi_1 & \Psi_2 \\ w_{,x}' & w_{,y}' \end{vmatrix} + \begin{vmatrix} -u & -v \\ F_{,x}' & F_{,y}' \end{vmatrix}$$

is a sum of terms of the same physical dimension. In contrast, the usual scalar product yields a dimensionally inconsistent sum.

Eqs. (35), (38a,b) and (39) are the transformation rules for the *values* of the vector and matrix functions. If they are to be applied to the matrix functions rather than to their values, then one has to keep in mind that the matrices associated with the original coordinates are functions of μ, whereas the starred matrices are functions of $\mu^* \equiv (\mu\cos\theta - \sin\theta)/(\cos\theta + \mu\sin\theta)$, because, under the coordinate transformation of Eq. (34), the complex variable $x + \mu y$ transforms into $\cos\theta + \mu\sin\theta$ multiplied by the new complex variable $x^* + \mu^* y^*$.

A key relationship between the matrix functions $\mathbf{J}(\mu)$ and $\mathbf{M}(\mu)$ may be verified by routine algebraic manipulation:

$$(\mu - \hat{\mu})[\![\mathbf{J}(\mu), \mathbf{J}(\hat{\mu})]\!] = \mathbf{M}(\mu) - \mathbf{M}(\hat{\mu}), \tag{40}$$

where the arguments μ and $\hat{\mu}$ may or may not be the same. Differentiation with respect to μ gives

$$(\mu - \hat{\mu})[\![\mathbf{J}'(\mu), \mathbf{J}(\hat{\mu})]\!] + [\![\mathbf{J}(\mu), \mathbf{J}(\hat{\mu})]\!] = \mathbf{M}'(\mu). \tag{41a}$$

Repeated differentiation yields, for $1 \leqslant s \leqslant N - 1$,

$$\begin{aligned} \partial_\mu^{N-s}\partial_{\hat{\mu}}^s\{(\mu - \hat{\mu})[\![\mathbf{J}(\mu), \mathbf{J}(\hat{\mu})]\!]\} &= (\mu - \hat{\mu})[\![\mathbf{J}^{(N-s)}(\mu), \mathbf{J}^{(s)}(\hat{\mu})]\!] + (N - s)[\![\mathbf{J}^{(N-s-1)}(\mu), \mathbf{J}^{(s)}(\hat{\mu})]\!] \\ &\quad - s[\![\mathbf{J}^{(N-s)}(\mu), \mathbf{J}^{(s-1)}(\hat{\mu})]\!] \\ &= \mathbf{0}. \end{aligned} \tag{41b}$$

For $\mu = \hat{\mu}$, Eqs. (41a) and repeated use of (41b) give

$$\begin{aligned} &[\![\mathbf{J}(\mu), \mathbf{J}(\mu)]\!] = \mathbf{M}'(\mu), \\ &[\![\mathbf{J}^{(N-s)}(\mu), \mathbf{J}^{(s)}(\mu)]\!] = \{s!(N - s)!/(N + 1)!\}\mathbf{M}^{(N+1)}(\mu) \quad (0 \leqslant s \leqslant N \leqslant 3). \end{aligned} \tag{42a, b}$$

Eqs. (22) and (42b) imply the following important expression for the binary product of any two eigenvectors of arbitrary orders $(0 \leqslant p, q \leqslant 3)$ sharing the same eigenvalue

$$\begin{aligned}[\![\boldsymbol{\xi}^{[p]}(\mu),\boldsymbol{\xi}^{[q]}(\mu)]\!] &= \sum_{0\leqslant k\leqslant p}\sum_{0\leqslant l\leqslant q}(p,k)(q,l)\{k!l!/(k+l+1)!\}(\boldsymbol{\eta}^{[p-k]})^{\mathrm{T}}\mathbf{M}^{(k+l+1)}\boldsymbol{\eta}^{[q-l]}\\ &= p!q!\sum_{q+1\leqslant s\leqslant p+q+1}\{1/(p+q+1-s)!s!\}(\boldsymbol{\eta}^{[p+q+1-s]})^{\mathrm{T}}\sum_{s-q\leqslant m\leqslant s}(s,m)\mathbf{M}^{(m)}\boldsymbol{\eta}^{[s-m]}.\end{aligned}\tag{43}$$

While the preceding expressions are always valid, repeated differentiation of Eq. (40) yields an equation which is valid only if μ and $\hat{\mu}$ are not equal:

$$\begin{aligned}\partial^p_\mu\partial^p_{\hat{\mu}}[\![\mathbf{J}(\mu),\mathbf{J}(\hat{\mu})]\!] &= (-1)^{q+1}\sum_{0\leqslant s\leqslant p}(p,s)(s+q)!(-\mu+\hat{\mu})^{-(s+q+1)}\mathbf{M}^{(p-s)}(\mu)\\ &\quad+(-1)^{p+1}\sum_{0\leqslant t\leqslant q}(q,t)(p+t)!(\mu-\hat{\mu})^{-(p+t+1)}\mathbf{M}^{(q-t)}(\hat{\mu}).\end{aligned}\tag{44}$$

If $\boldsymbol{\xi}^{[p]}(\mu)$ and $\boldsymbol{\xi}^{[q]}(\hat{\mu})$ are eigenvectors of orders p and q respectively $(0\leqslant p,q\leqslant 3)$, associated with two *distinct* eigenvalues μ and $\hat{\mu}$, then Eqs. (22) and (23) yield

$$\begin{aligned}&[\![\boldsymbol{\xi}^{[p]}(\mu),\boldsymbol{\xi}^{[q]}(\hat{\mu})]\!]\\ &= \sum_{0\leqslant s\leqslant p}\sum_{0\leqslant t\leqslant q}(p,s)(q,t)\boldsymbol{\eta}^{[p-s]}(\mu)^{\mathrm{T}}[\![\mathbf{J}^{(s)}(\mu),\mathbf{J}^{(t)}(\hat{\mu})]\!]\boldsymbol{\eta}^{[q-t]}(\hat{\mu})\\ &= \sum_{0\leqslant s\leqslant p}\sum_{0\leqslant t\leqslant q}(-1)^{t+1}\sum_{0\leqslant m\leqslant s}(p,s)(q,t)(s,m)(m+t)!(-\mu+\hat{\mu})^{-(t+m+1)}\boldsymbol{\eta}^{[p-s]}(\mu)^{\mathrm{T}}\,\mathbf{M}^{(s-m)}(\mu)\boldsymbol{\eta}^{[q-t]}(\hat{\mu})\\ &\quad+\sum_{0\leqslant s\leqslant p}\sum_{0\leqslant t\leqslant q}(-1)^{s+1}\sum_{0\leqslant m\leqslant t}(p,s)(q,t)(t,m)(m+s)!(\mu-\hat{\mu})^{-(s+m+1)}\boldsymbol{\eta}^{[p-s]}(\mu)^{\mathrm{T}}\mathbf{M}^{(t-m)}(\hat{\mu})\boldsymbol{\eta}^{[q-t]}(\hat{\mu})\\ &= \sum_{0\leqslant t\leqslant q}\sum_{0\leqslant m\leqslant p}(-1)^{t+1}(q,t)(p,m)(t+m)!(-\mu+\hat{\mu})^{-(t+m+1)}\boldsymbol{\eta}^{[q-t]}(\hat{\mu})^{\mathrm{T}}\sum_{m\leqslant k\leqslant p}(p-m,k-m)\mathbf{M}^{(k-m)}(\mu)\boldsymbol{\eta}^{[p-k]}(\mu)\\ &\quad+\sum_{0\leqslant s\leqslant p}\sum_{0\leqslant m\leqslant q}(-1)^{s+1}(p,s)(q,m)(s+m)!(\mu-\hat{\mu})^{-(s+m+1)}\boldsymbol{\eta}^{[p-s]}(\mu)^{\mathrm{T}}\\ &\quad\times\sum_{m\leqslant k\leqslant q}(q-m,k-m)\mathbf{M}^{(k-m)}(\hat{\mu})\boldsymbol{\eta}^{[q-k]}(\mu)=\mathbf{0}.\end{aligned}\tag{45}$$

Notice that the two innermost sums in the last expression vanish in view of Eq. (23). Eq. (45) shows the *orthogonality* of the eigenvectors associated with distinct eigenvalues. In the terminology of vector space, the eight-dimensional solution space is the direct sum of a number of eigenspaces that are mutually *orthogonal* in the sense of the binary product. Each eigenspace is spanned by the eigenvectors of various orders associated with a single distinct eigenvalue, whose multiplicity equals the dimension of that eigenspace.

Orthogonality of the *zeroth-order* eigenvectors follows easily from our Eqs. (18) and (40). It was shown by Stroh (1958) using a different proof

The orthogonality relation (45) implies that

$$[\![\mathbf{Z}^+,\overline{\mathbf{Z}}^+]\!]=[\![\overline{\mathbf{Z}}^+,\mathbf{Z}^+]\!]=\mathbf{0}.\tag{46}$$

Eqs. (39) and (46) yield

$$[\![\mathbf{Z},\mathbf{Z}]\!]=[\![\{\mathbf{Z}^+,\overline{\mathbf{Z}}^+\},\{\mathbf{Z}^+,\overline{\mathbf{Z}}^+\}]\!]=\begin{bmatrix}\boldsymbol{\Omega} & 0\\ 0 & \overline{\boldsymbol{\Omega}}\end{bmatrix}.\tag{47}$$

Consequently,

$$(\mathrm{Det}[\mathbf{Z}])^2=\mathrm{Det}[\boldsymbol{\Omega}]\mathrm{Det}[\overline{\boldsymbol{\Omega}}].\tag{48}$$

The orthogonality relation (45) also implies that the symmetric matrix $\mathbf{\Omega}$ is in the block diagonal form, where each block $\boldsymbol{\omega}_k$ is associated with a distinct eigenvalue μ_k. $\boldsymbol{\omega}_k$ is the pseudo-metric of the eigenspace of μ_k. If μ_k has the multiplicity p, we let its p independent eigenvectors be arranged in ascending orders and combined into an $8 \times p$ matrix $\mathbf{X}_k$. Then

$$\boldsymbol{\omega}_k = [\![\mathbf{X}_k, \mathbf{X}_k]\!], \tag{49}$$

$$\mathbf{\Omega} = \langle \boldsymbol{\omega}_k \rangle, \tag{50}$$

and Eq. (48) becomes

$$(\mathrm{Det}[\mathbf{Z}])^2 = \prod_k |\mathrm{Det}[\boldsymbol{\omega}_k]|^2. \tag{51}$$

In the next section, the pseudo-metrics $\boldsymbol{\omega}_k$ will be explicitly obtained for the eigenspaces of all eight types of normal, abnormal and superabnormal eigenvalues, and their inverse matrices will be found analytically. Hence the matrices $\mathbf{\Omega}$ and $\mathbf{Z}$ are also nonsingular, and they have unique inverses $\mathbf{\Omega}^{-1}$ and $\mathbf{Z}^{-1}$. Notice that the invertibility of $\mathbf{Z}$ implies that the eight eigenvectors and the associated eigensolutions are indeed independent. This has been previously assumed but has never been proved for the various degenerate cases.

4. Pseudo-metrics of the various types of eigenspaces

It was pointed out in the previous section that there are eight distinct types of eigenvalues depending on multiplicity and degeneracy or nondegeneracy. To each type of eigenvalue is associated a type of eigenspace, spanned by a set of independent eigenvectors of various orders, whose dimension equals the multiplicity of the eigenvalue.

The eigenvectors that span the p-dimensional eigenspace of a multiple eigenvalue μ are generally *not* orthogonal. For two eigenvectors sharing a common *normal* eigenvalue, substitution of solution (28) into Eq. (43) yields the important expression of the binary product

$$\begin{aligned}[\![\boldsymbol{\xi}^{[i-1]}, \boldsymbol{\xi}^{[j-1]}]\!] &= (W\delta)^{(i+j-1)}(i-1)!(j-1)!/(i+j-1)! \\ &= \{(i-1)!(j-1)!/(i+j-1)!\}\{\delta^{(i+j-1)}W + (i+j-1,1)\delta^{(i+j-2)}W' \\ &\quad + (i+j-1,2)\delta^{(i+j-3)}W'' + (i+j-1,3)\delta^{(i+j-4)}W'''\},\end{aligned} \tag{52}$$

where negative-order derivatives are taken to be zero.

Eq. (52) yields the pseudo-metrics $\boldsymbol{\omega}_{[N1]}$, $\boldsymbol{\omega}_{[N2]}$, $\boldsymbol{\omega}_{[N3]}$ and $\boldsymbol{\omega}_{[N4]}$ of the eigenspaces associated with a normal eigenvalue of multiplicity from one to four. The inverses of these matrices may also be obtained. The results are given in Eqs. (A.1)–(A.4) of Appendix A. These equations are formally identical to the corresponding results in plane anisotropy given by Eqs. (4.13), (4.14), (5.13a) and (5.13b) in Yin (2000), even though the functions δ and l_2 in the latter work are polynomials of degrees six and two, respectively, while δ and W in the present analysis are polynomials of higher degrees.

We next consider the eigenspace associated with an abnormal eigenvalue μ, for which $\mathbf{M}(\mu)$, and hence also $\mathbf{W}(\mu)$, are zero matrices while $\mathbf{M}'(\mu)$ and $\mathbf{W}'(\mu)$ are not. There are two independent zeroth-order eigenvectors given by the two columns of $\mathbf{J}(\mu)$. For $N = 2$, Eq. (26) reduces to

$$2\mathbf{M}'(\mu)\mathbf{W}'(\mu) = \delta''(\mu)\mathbf{I}. \tag{53}$$

If μ is of multiplicity two, then $\delta''(\mu) \neq 0$ and therefore $\mathrm{Det}[\mathbf{M}'(\mu)]$ cannot vanish. According to Eq. (42a), the two-dimensional eigenspace has the pseudo-metric

$$\boldsymbol{\omega}_{[A2]} = [\![\mathbf{J}(\mu), \mathbf{J}(\mu)]\!] = \mathbf{M}'(\mu). \tag{54a}$$

Eq. (53) gives the inverse matrix

$$\boldsymbol{\omega}_{[A2]}^{-1} = (2/\delta'')\mathbf{W}'(\mu). \tag{54b}$$

If μ_0 is a triple abnormal eigenvalue, then three eigenvectors may be chosen as follows to ensure their linear independence:

$$\boldsymbol{\xi}^{[0]} = \mathbf{J}(\mu_0)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\hat{\boldsymbol{\rho}}, \quad \boldsymbol{\xi}^{[1]} = (\mathbf{JW})'\hat{\boldsymbol{\rho}} = \mathbf{JW}'\hat{\boldsymbol{\rho}}, \quad \boldsymbol{\xi}^{[2]} = (\mathbf{JW})''\hat{\boldsymbol{\rho}} = (\mathbf{JW}'' + 2\mathbf{J}'\mathbf{W}')\hat{\boldsymbol{\rho}}, \tag{55a, b, c}$$

where $\hat{\boldsymbol{\rho}}$ is the column selector defined by Eq. (28a). For an abnormal quadruple eigenvalue one has, in addition,

$$\boldsymbol{\xi}^{[3]} = (\mathbf{JW})'''\hat{\boldsymbol{\rho}} = (\mathbf{JW}''' + 3\mathbf{J}'\mathbf{W}'' + 3\mathbf{J}''\mathbf{W}')\hat{\boldsymbol{\rho}}. \tag{55d}$$

Using Eqs. (55a–c) and (53) with $\delta''(\mu_0) = 0$, one obtains the pseudo-metric $\boldsymbol{\omega}_{[A3]}$ of the eigenspace associated with a triple abnormal eigenvalue. $\boldsymbol{\omega}_{[A3]}$ and its inverse matrix are given in Eq. (A.7).

Once again, Eqs. (54a,b), (55a–c) and Eq. (A.7) are formally identical to the corresponding results in plane anisotropic elasticity (Yin, 2000), even though the matrices and eigenvectors in the present problem have higher dimensions. However, the results for a triple abnormal eigenvalue, as given by Eqs. (4.18), (4.19) and (4.20a,b) in Yin (2000), contain errors. The errors result from missing a factor 2 for the term containing $\mathbf{J}'$ and $\mathbf{K}'$ in Eqs. (4.18) and (4.19). The two equations should be replaced by the present Eq. (55c). In addition, Eqs. (4.20a,b) in that paper should be replaced by present Eqs. (A.7a,b), where W is to be changed into l_2. However, the matrices **L**, **H** and **S**, as given by Eq. (4.21) of the paper, are correct and formally in agreement with the present results for laminated plates with a triple abnormal eigenvalue.

If μ is an abnormal quadruple eigenvalue, then one obtains the pseudo-metric $\boldsymbol{\Omega} = \boldsymbol{\omega}_{[A4]}$ from Eqs. (55a–d) and $\delta(\mu) = \delta'(\mu) = \delta''(\mu) = \delta'''(\mu) = 0$. The result and the inverse matrix are given in Eq. (A.8) of Appendix A.

For a superabnormal eigenvalue μ_0, both $\mathbf{M}(\mu_0)$ and $\mathbf{M}'(\mu_0)$ are null matrices. The governing equation (23) imposes no restriction on $\boldsymbol{\eta}^{[0]}$ and $\boldsymbol{\eta}^{[1]}$. Four eigenvectors are obtained by using the columns of the matrices $\mathbf{J}(\mu_0)$ and $\mathbf{J}'(\mu_0)$. This yields the pseudo-metric

$$\boldsymbol{\omega}_{[SA]} = [\![\{\mathbf{J}, \mathbf{J}'\}, \{\mathbf{J}, \mathbf{J}'\}]\!] = \begin{bmatrix} \mathbf{0}_{2\times 2} & (1/2)\mathbf{M}''(\mu_0) \\ (1/2)\mathbf{M}''(\mu_0) & (1/6)\mathbf{M}'''(\mu_0) \end{bmatrix}, \tag{56a}$$

whose inverse is

$$\boldsymbol{\omega}_{[SA]}^{-1} = (12/\delta'''')\begin{bmatrix} -\mathbf{W}'''/3 & \mathbf{W}'' \\ \mathbf{W}'' & \mathbf{0}_{2\times 2} \end{bmatrix}. \tag{56b}$$

Notice that the matrix product of (56a) and (56b) yields the identity matrix provided that

$$6\mathbf{M}''\mathbf{W}'' = \delta''''\mathbf{I}, \quad \mathbf{M}'''\mathbf{W}'' - \mathbf{M}''W''' = \mathbf{0}.$$

The first equality is implied by $\mathbf{M} = \mathbf{M}' = \mathbf{W} = \mathbf{W}' = \mathbf{0}$ and Eq. (25) with $N = 4$. The second equality easily follows from the following expressions for a superabnormal μ_0

$$\mathbf{M}(\mu) = (\mu - \mu_0)^2(\mu - \bar{\mu}_0)^2\mathbf{C}, \quad \mathbf{W}(\mu) = (\mu - \mu_0)^2(\mu - \bar{\mu}_0)^2\mathbf{C}', \tag{57a, b}$$

where

$$\mathbf{C} = |\mu_0|^{-4}\begin{bmatrix} D_{22}^* & B_{22}^* \\ B_{22}^* & -A_{22}^* \end{bmatrix}, \quad \mathbf{C}' = |\mu_0|^{-4}\begin{bmatrix} -A_{22}^* & -B_{22}^* \\ -B_{22}^* & D_{22}^* \end{bmatrix}. \tag{57c, d}$$

5. Projection operators and related intrinsic tensors

It was shown in the preceding two sections that, for each distinct eigenvalue μ_k of multiplicity p ($1 \leqslant p \leqslant 4$), the p-dimensional eigenspace has a *nonsingular* pseudo-metric $\boldsymbol{\omega}_k = [\![\mathbf{X}_k, \mathbf{X}_k]\!]$ referred to the eigenvectors in $\mathbf{X}_k$. Consider the 8×8 symmetric matrix

$$\mathbf{X}_k \boldsymbol{\omega}_k^{-1} \mathbf{X}_k^{\mathrm{T}} = \mathbf{F}_k + \mathrm{i}\mathbf{G}_k, \tag{58}$$

where $\mathbf{F}_k$ and $\mathbf{G}_k$ are *real*, *symmetric* matrices. Postmultiplying by $\mathbf{IIX}_k$, and using Eq. (49), one obtains

$$\mathbf{X}_k = (\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{IIX}_k. \tag{59}$$

Therefore, the linear transformation $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}$ maps every vector in the eigenspace of μ_k into itself. On the other hand, if $\boldsymbol{\xi}^*$ is an eigenvector associated with a different eigenvalue, then it is orthogonal to all columns of $\mathbf{X}_k$, and Eq. (58) yields $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}\boldsymbol{\xi}^* = \mathbf{0}$. Therefore, $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}$ is the *projection operator* into the eigenspace of μ_k, i.e., any eight-dimensional vector $\mathbf{v}$ may be decomposed as $\mathbf{v} = (\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{IIv} + \mathbf{v}'$, where the first part belongs to the eigenspace of μ_k, and $\mathbf{v}'$ is orthogonal to that eigenspace. For the conjugate eigenvalue $\bar{\mu}_k$, Eqs. (49) and (58) are replaced by their complex conjugates, and one finds that the projection operator is given by $(\mathbf{F}_k - \mathrm{i}\mathbf{G}_k)\mathbf{II}$. This yields the decomposition of the identity transformation into orthogonal projections:

$$\mathbf{I}_8 = \sum_k (\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II} + \sum_k (\mathbf{F}_k - \mathrm{i}\mathbf{G}_k)\mathbf{II} = 2\left(\sum_k \mathbf{F}_k\right)\mathbf{II}, \tag{60}$$

or, equivalently,

$$2\sum_k \mathbf{F}_k = \mathbf{II},$$

where the summation extends over all eigenvalues with $\mathrm{Im}[\mu] > 0$.

A similar argument, applied to the operators $\mu_k(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}$ and $\bar{\mu}_k(\mathbf{F}_k - \mathrm{i}\mathbf{G}_k)\mathbf{II}$, yields the following equations instead of the last two:

$$\mathbf{TIIX}_k = \mu_k \mathbf{X}_k, \quad \mathbf{TII}\overline{\mathbf{X}}_k = \bar{\mu}_k \overline{\mathbf{X}}_k, \tag{61}$$

$$\mathbf{T} \equiv \sum_k \mu_k(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k) + \sum_k \bar{\mu}_k(\mathbf{F}_k - \mathrm{i}\mathbf{G}_k). \tag{62}$$

Obviously, $\mathbf{T}$ is a real, symmetric matrix. Eq. (61) shows that the real matrix $\mathbf{TII}$ has its eigenvalues and eigenvectors coinciding with the laminate eigenvalues and eigenvectors. Let $\langle \mu \rangle$ denote the diagonal matrix of the eight eigenvalues arranged in the same order as the associated eigenvectors in $\mathbf{Z}$. Then, substitution of Eq. (58) and its complex conjugate into Eq. (62) yields

$$\mathbf{TII} = \mathbf{Z}\langle \mu \rangle \langle \boldsymbol{\Omega}^{-1}, \overline{\boldsymbol{\Omega}}^{-1} \rangle \mathbf{Z}^{\mathrm{T}}\mathbf{II} = \mathbf{Z}\langle \mu \rangle \mathbf{Z}^{-1}. \tag{63}$$

When one projection is followed by another, the effect is the null transformation except when the two projections are identical. Hence $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}(\mathbf{F}_k - \mathrm{i}\mathbf{G}_k) = \mathbf{0}$ and, for $k \neq j$,

$$(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}(\mathbf{F}_j + \mathrm{i}\mathbf{G}_j) = \mathbf{0}, \quad (\mathbf{F}_k - \mathrm{i}\mathbf{G}_k)\mathbf{II}(\mathbf{F}_j + \mathrm{i}\mathbf{G}_j) = \mathbf{0}.$$

Separating into real and imaginary parts, one obtains

$$\begin{aligned} &\mathbf{F}_k\mathbf{IIF}_k + \mathbf{G}_k\mathbf{IIG}_k = \mathbf{G}_k\mathbf{IIF}_k - \mathbf{F}_k\mathbf{IIG}_k = \mathbf{0}, \\ &\mathbf{F}_k\mathbf{IIF}_j = \mathbf{F}_k\mathbf{IIG}_j = \mathbf{G}_k\mathbf{IIF}_j = \mathbf{G}_k\mathbf{IIG}_j = \mathbf{0} \quad \text{for } k \neq j. \end{aligned} \tag{64}$$

When a projection is repeated, the effect is the same as applying the projection only once. Hence

$$\mathbf{F}_k\mathbf{I I}\mathbf{F}_k - \mathbf{G}_k\mathbf{I I}\mathbf{G}_k = \mathbf{F}_k, \quad \mathbf{G}_k\mathbf{I I}\mathbf{F}_k + \mathbf{F}_k\mathbf{I I}\mathbf{G}_k = \mathbf{G}_k.$$

It follows that

$$\mathbf{F}_k\mathbf{I I}\mathbf{F}_k = -\mathbf{G}_k\mathbf{I I}\mathbf{G}_k = (1/2)\mathbf{F}_k, \quad \mathbf{G}_k\mathbf{I I}\mathbf{F}_k = \mathbf{F}_k\mathbf{I I}\mathbf{G}_k = (1/2)\mathbf{G}_k. \tag{65a, b}$$

Summing Eq. (58) over all eigenvalues with positive imaginary parts, and then do the same for all eigenvalues with negative imaginary parts, one obtains

$$(1/2)(\mathbf{F} + \mathrm{i}\mathbf{G}) \equiv \mathbf{Z}^+\mathbf{\Omega}^{-1}(\mathbf{Z}^+)^{\mathrm{T}} = \sum_k \mathbf{F}_k + \mathrm{i}\sum_k \mathbf{G}_k, \tag{66}$$

as well as the complex conjugate of (66). Hence,

$$\mathbf{F} = 2\sum_k \mathbf{F}_k = \mathbf{I I}, \quad \mathbf{G} = 2\sum_k \mathbf{G}_k, \tag{67a, b}$$

where the summations also extend over all eigenvalues with $\mathrm{Im}[\mu] > 0$. Eq. (66) and its complex conjugate imply

$$(1/2)(\mathbf{I} + \mathrm{i}\mathbf{G}\mathbf{I I})\mathbf{Z}^+ \equiv \mathbf{Z}^+, \quad (1/2)(\mathbf{I} + \mathrm{i}\mathbf{G}\mathbf{I I})\overline{\mathbf{Z}}^+ \equiv \mathbf{0}, \tag{68a, b}$$

$$(1/2)(\mathbf{I} - \mathrm{i}\mathbf{G}\mathbf{I I})\overline{\mathbf{Z}}^+ \equiv \overline{\mathbf{Z}}^+, \quad (1/2)(\mathbf{I} - \mathrm{i}\mathbf{G}\mathbf{I I})\mathbf{Z}^+ \equiv \mathbf{0}. \tag{68c, d}$$

Hence $(1/2)(\mathbf{I} + \mathrm{i}\mathbf{G}\mathbf{I I})$ and $(1/2)(\mathbf{I} - \mathrm{i}\mathbf{G}\mathbf{I I})$ are, respectively, the projection operators from the solution space to the subspace spanned by $\mathbf{Z}^+$ and to its conjugate subspace. These relations also yield

$$\mathbf{G}\mathbf{I I}\mathbf{Z}^+ \equiv -\mathrm{i}\mathbf{Z}^+, \quad \mathbf{G}\mathbf{I I}\overline{\mathbf{Z}}^+ \equiv \mathrm{i}\overline{\mathbf{Z}}^+, \tag{69a, b}$$

i.e., the matrix $\mathbf{G}\mathbf{I I}$ has $-\mathrm{i}$ and $+\mathrm{i}$ as quadruple eigenvalues and the corresponding eigenvectors are, respectively, the four columns of $\mathbf{Z}^+$ and $\overline{\mathbf{Z}}^+$:

$$\mathbf{G}\mathbf{I I}\mathbf{Z} = \mathbf{Z}\langle -\mathrm{i}\mathbf{I}_4, \mathrm{i}\mathbf{I}_4\rangle. \tag{69c}$$

Notice also that Eq. (62) and (64a,b) yield the following and their complex conjugate equalities

$$\mathbf{T}\mathbf{I I}(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k) = (\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{I I}\mathbf{T} = \mu_k(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k).$$

Eq. (65a) gives $\mathbf{F}_k = -2\mathbf{G}_k\mathbf{I I}\mathbf{G}_k$. Substitution into Eq. (60) yields $-4\sum_k \mathbf{G}_k\mathbf{I I}\mathbf{G}_k = \mathbf{I I}$. Using the orthogonality relations of Eq. (64) for $k \neq j$, one obtains

$$\mathbf{G}\mathbf{I I}\mathbf{G}\mathbf{I I} = -\mathbf{I}_8. \tag{70}$$

We define the 4×4 submatrices $\mathbf{L}$, $\mathbf{S}$, $\mathbf{H}$ of $\mathbf{G}$, and the submatrices $\mathbf{L}_k$, $\mathbf{S}_k$ and $\mathbf{H}_k$ of $\mathbf{G}_k$:

$$\mathbf{G} \equiv \begin{bmatrix} -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \\ \mathbf{S} & \mathbf{H} \end{bmatrix} \equiv 2\sum_k \mathbf{G}_k = 2\sum_k \begin{bmatrix} -\mathbf{L}_k & \mathbf{S}_k^{\mathrm{T}} \\ \mathbf{S}_k & \mathbf{H}_k \end{bmatrix}. \tag{71}$$

Each eigenspace $\mathbf{X}_k$ contributes the term $\mathbf{G}_k$ to Eq. (71), and the conjugate space $\overline{\mathbf{X}}_k$ contributes the same. This accounts for the factor 2 in the last two expressions of (71). Symmetry of $\mathbf{G}_k$ implies that $\mathbf{L}_k$, $\mathbf{H}_k$, $\mathbf{L}$ and $\mathbf{H}$ are symmetric, while $\mathbf{S}_k$ and $\mathbf{S}$ are generally not symmetric. Eq. (70) implies that

$$\mathbf{G}^{-1} = -\mathbf{I I}\mathbf{G}\mathbf{I I},$$

$$\mathbf{H}\mathbf{L} - \mathbf{S}\mathbf{S} = \mathbf{L}\mathbf{H} - \mathbf{S}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} = \mathbf{I}_4, \tag{72}$$

$$\mathbf{L}\mathbf{S} = -(\mathbf{L}\mathbf{S})^{\mathrm{T}}, \quad \mathbf{S}\mathbf{H} = -(\mathbf{S}\mathbf{H})^{\mathrm{T}}. \tag{73}$$

Notice that $\mathbf{\Omega}$ is composed of diagonal blocks $\boldsymbol{\omega}_k$, and $\mathbf{Z}^+$ is formed by joining the corresponding submatrices $\mathbf{X}_k$. By summing Eq. (58) over k, using Eq. (67), and doing the same for the complex conjugate of (58), one obtains

$$\mathbf{Z}\langle\mathbf{\Omega}^{-1}, \overline{\mathbf{\Omega}}^{-1}\rangle\mathbf{Z}^{\mathrm{T}} = 2\sum_k \mathbf{F}_k = \mathbf{II}, \tag{74}$$

$$\mathbf{Z}\langle\mathbf{\Omega}^{-1}, -\overline{\mathbf{\Omega}}^{-1}\rangle\mathbf{Z}^{\mathrm{T}} = 2\mathrm{i}\sum_k \mathbf{G}_k = \mathrm{i}\mathbf{G}. \tag{75}$$

When the last equation is postmultiplied by $-\mathrm{i}\mathbf{II}$, it yields the following expression which allows $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{S}$ to be calculated directly from all eigenvectors and the reciprocal base vectors without first obtaining $\mathbf{\Omega}$:

$$\mathbf{GII} = -\mathrm{i}\mathbf{Z}\langle\mathbf{\Omega}^{-1}, -\overline{\mathbf{\Omega}}^{-1}\rangle\mathbf{Z}^{\mathrm{T}}\mathbf{IIZZ}^{-1} = \mathbf{Z}\langle -\mathrm{i}\mathbf{I}_4, \mathrm{i}\mathbf{I}_4\rangle\mathbf{Z}^{-1}. \tag{76}$$

This remarkable simple expression also follows directly from Eq. (69c). Notice that Eqs. (74) and (75) may be combined to yield Eq. (66) and its complex conjugate.

Under the coordinate transformation of Eq. (34), $\mathbf{\Omega}$ is unchanged according to Eq. (39), while $\mathbf{Z}^+$ and its complex conjugate transform according to Eq. (35). Hence the left-hand side of Eq. (75), and therefore also the matrix $\mathbf{G}$, has to be premultiplied by $\mathbf{Q}_8$ and postmultiplied by $\mathbf{Q}_8^{\mathrm{T}}$ to obtain the image matrices after the rotation of coordinates. Consequently, the real matrices $\mathbf{G}$, $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{S}$ conform to the tensorial transformation rule

$$\mathbf{G}^* = \mathbf{Q}_8\mathbf{G}\mathbf{Q}_8^{\mathrm{T}}, \quad \mathbf{L}^* = \mathbf{Q}_4\mathbf{L}\mathbf{Q}_4^{\mathrm{T}}, \quad \mathbf{H}^* = \mathbf{Q}_4\mathbf{H}\mathbf{Q}_4^{\mathrm{T}}, \quad \mathbf{S}^* = \mathbf{Q}_4\mathbf{S}\mathbf{Q}_4^{\mathrm{T}}, \tag{77}$$

where $\mathbf{Q}_4 \equiv \langle\mathbf{Q}_2, \mathbf{Q}_2\rangle$.

The Stroh–Barnett–Lothe tensors in 2-D anisotropic elasticity satisfy equations identical in form to (72) and (73), and other equations obtained by substituting the first equation of (71) into (74)–(77), except that these tensors have the dimension three. Now the same identities have been established for all anisotropic laminates, for which $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{S}$ are 4×4 real matrices.

Let τ_k be a *nonsingular*, complex linear transformation in the eigenspace of μ_k. Then it maps the column vectors of $\mathbf{X}_k$ into the column vectors that form another matrix $\mathbf{X}_k^* = \mathbf{X}_k\tau_k$. Consider

$$\boldsymbol{\omega}_k^* \equiv [\![\mathbf{X}_k^*, \mathbf{X}_k^*]\!] = \tau_k^{\mathrm{T}}[\mathbf{X}_k, \mathbf{X}_k]\tau_k = \tau_k^{\mathrm{T}}\boldsymbol{\omega}_k\tau_k,$$

$\boldsymbol{\omega}_k^*$ has the inverse matrix which satisfy

$$\mathbf{X}_k^*\boldsymbol{\omega}_k^{*-1}\mathbf{X}_k^* = (\mathbf{X}_k\tau_k)\{\tau_k^{-1}\boldsymbol{\omega}_k^{-1}(\tau_k^{\mathrm{T}})^{-1}\}(\tau_k^{\mathrm{T}}\mathbf{X}_k^{\mathrm{T}}) = \mathbf{X}_k\boldsymbol{\omega}_k^{-1}\mathbf{X}_k = \mathbf{F}_k + \mathrm{i}\mathbf{G}_k. \tag{78}$$

That is, $\mathbf{F}_k + \mathrm{i}\mathbf{G}_k$ and the projection operator $(\mathbf{F}_k + \mathrm{i}\mathbf{G}_k)\mathbf{II}$ are invariant under an affine transformation of the base vectors of the eigenspace from the set $\mathbf{X}_k$ to another set $\mathbf{X}_k^*$.

Consider, similarly, a nonsingular linear transformation, $\tau : \mathbf{Z}^+ \to (\mathbf{Z}^+)^* = \mathbf{Z}^+\tau$, in the four-dimensional space spanned by the vectors of $\mathbf{Z}^+$, and the conjugate transformation $\bar{\tau} : \bar{\mathbf{Z}}^+ \to (\bar{\mathbf{Z}}^+)^* = \mathbf{Z}^+\bar{\tau}$. Eq. (77) yields

$$\mathbf{Z}^*\begin{bmatrix} -\mathrm{i}\mathbf{I}_4 & \mathbf{0} \\ \mathbf{0} & \mathrm{i}\mathbf{I}_4 \end{bmatrix}(\mathbf{Z}^*)^{-1} = \begin{bmatrix} \mathbf{S}^{\mathrm{T}} & -\mathbf{L} \\ \mathbf{H} & \mathbf{S} \end{bmatrix}, \tag{79}$$

where $\mathbf{Z}^* = \{(\mathbf{Z}^+)^*, (\bar{\mathbf{Z}}^+)^*\}$. Hence $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{S}$ are invariant under an arbitrary affine transformation of the base vectors that preserves the complex conjugate relation. In other words, these matrices can be calculated from Eq. (79) using any set of four linearly independent eigenvectors and their complex conjugates. The results are not different. (On the other hand, different choices of base vectors yield different pseudo-metrics, and the simple forms given in Appendix A are obtained only if the higher-order eigenvectors are related to the lower-order ones according to the derivative rule.) There is neither an advantage nor a need to normalize the eigenvectors. Normalization aims to find a set of eigenvectors whose scalar or binary product results in the identity matrix. This is generally impossible when the laminate is degenerate, extra-degenerate, or ultra-degenerate. Certain lower-order eigenvectors $\boldsymbol{\xi}$ associated with a triple or quadruple eigenvalue are found to satisfy $[\![\boldsymbol{\xi},\boldsymbol{\xi}]\!] = 0$ For example, Eq. (A.4a) shows that the two lowest-order eigenvectors of a quadruple normal eigenvalue yield $\Omega_{11} = \Omega_{22} = 0$. Such eigenvectors cannot be normalized. Even for eigenvectors with $[\![\boldsymbol{\xi},\boldsymbol{\xi}]\!] \neq 0$, normalization generally results in complicated analytical forms of expression, which make the implementation of the derivative rule unduly cumbersome. The present remarks concerning normalization of eigenvectors apply also to the case of two-dimensional elasticity. Although normalization has not found a role in the Lekhnitskii formalism, it has often been adopted in the ERSS formalism of the nondegenerate cases with no apparent benefit.

Since the binary product is complex-valued, it does not define a metric in the vector space of solutions. Hence it cannot endow the latter with a Euclidean geometrical structure. Orthogonality and projections are valid concepts associated with the binary product in a complex vector space, but normalization and unit vectors require a length measure which is not provided for all vectors in an eigenspace associated with a multiple eigenvalue. Not only is normalization an alien concept, it is also not needed since the physical solutions and physical entities such as **L**, **H** and **S**, given by Eqs. (71) and (75) or (79), are invariant under any affine transformation of the base vectors.

For the eigenspace associated with a *normal* eigenvalue, the eigenvectors may indeed be normalized in a special way (but not in the sense of requiring various eigenvectors to have "unit magnitudes"), so that the resulting pseudo-metric $\boldsymbol{\omega}_k$ depends only on the derivatives of δ and not on W as defined in Eq. (27b). Instead of using the eigenvectors $\boldsymbol{\xi}^{[j]} = (\mathrm{d}^j/\mathrm{d}\mu^j)\ (\mathbf{JW})\boldsymbol{\rho}$, one may use $\boldsymbol{\xi}^{[j]} = (\mathrm{d}^j/\mathrm{d}\mu^j)(\mathbf{JW}/\sqrt{W})\boldsymbol{\rho}$. Then, instead of Eq. (52), one has the following simple expression for the components of the pseudo-metric:

$$\boldsymbol{\omega}_{ij} = [\![\boldsymbol{\xi}^{[i-1]},\boldsymbol{\xi}^{[j-1]}]\!] = \delta^{(i+j-1)}(i-1)!(j-1)!/(i+j-1)!. \tag{80}$$

This pseudo-metric depends only on the various derivatives of δ of order greater than $p-1$ (since the lower-order derivatives vanish). For an *abnormal* eigenvalue, it does not seem feasible to normalize the eigenvectors so that the resulting pseudo-metric depends only on the derivatives of δ.

The pseudo-metric $\langle\boldsymbol{\Omega},\bar{\boldsymbol{\Omega}}\rangle$ of the eight-dimensional solution space has the inverse matrix $\langle\boldsymbol{\Omega}^{-1},\bar{\boldsymbol{\Omega}}^{-1}\rangle$. The latter is the pseudo-metric of the dual space spanned by the reciprocal base vectors. The reciprocal base vectors are the columns of the 8×8 matrix $\mathbf{Y} = (\mathbf{Z}^{-1})^{\mathrm{T}}$ (henceforth dissociated from the previous definition of **Y** given by Eq. (10a)). One has

$$\mathbf{Z}^{\mathrm{T}}\mathbf{Y} = \mathbf{Y}^{\mathrm{T}}\mathbf{Z} = \mathbf{Y}\mathbf{Z}^{\mathrm{T}} = \mathbf{Z}\mathbf{Y}^{\mathrm{T}} = \mathbf{I}_8. \tag{81}$$

From every relationship valid for the original base vectors, a corresponding relationship may be easily derived for the dual base vectors. Some useful relations may be paired as follows

$$\mathbf{Z}^{\mathrm{T}}\mathbf{II}\mathbf{Z} = \langle\boldsymbol{\Omega},\bar{\boldsymbol{\Omega}}\rangle, \quad \mathbf{Y}^{\mathrm{T}}\mathbf{II}\mathbf{Y} = \langle\boldsymbol{\Omega}^{-1},\bar{\boldsymbol{\Omega}}^{-1}\rangle, \tag{82}$$

$$\mathbf{Y} = \mathbf{II}\mathbf{Z}\langle\boldsymbol{\Omega}^{-1},\bar{\boldsymbol{\Omega}}^{-1}\rangle, \quad \mathbf{Z} = \mathbf{II}\mathbf{Y}\langle\boldsymbol{\Omega},\bar{\boldsymbol{\Omega}}\rangle, \tag{83}$$

$$\mathbf{Z}\langle\boldsymbol{\Omega}^{-1},\bar{\boldsymbol{\Omega}}^{-1}\rangle\mathbf{Z}^{\mathrm{T}} = \mathbf{II}, \quad \mathbf{Y}\langle\boldsymbol{\Omega},\bar{\boldsymbol{\Omega}}\rangle\mathbf{Y}^{\mathrm{T}} = \mathbf{II}, \tag{84}$$

$$\mathbf{Z}\langle\boldsymbol{\Omega}^{-1},-\bar{\boldsymbol{\Omega}}^{-1}\rangle\mathbf{Z}^{\mathrm{T}} = \mathrm{i}\mathbf{G}, \quad \mathbf{Y}\langle\boldsymbol{\Omega},-\bar{\boldsymbol{\Omega}}\rangle\mathbf{Y}^{\mathrm{T}} = \mathrm{i}\mathbf{II}\mathbf{G}\mathbf{II}, \tag{85}$$

$$\mathbf{Z}\langle\mathbf{I}_4,-\mathbf{I}_4\rangle\mathbf{Z}^{-1} = \mathrm{i}\mathbf{G}\mathbf{II}, \quad \mathbf{Y}\langle\mathbf{I}_4,-\mathbf{I}_4\rangle\mathbf{Y}^{-1} = \mathrm{i}\mathbf{II}\mathbf{G}, \tag{86}$$

$$\mathbf{T}\mathbf{II}\mathbf{Z} = \mathbf{Z}\langle\boldsymbol{\mu}\rangle, \quad \mathbf{T}^{-1}\mathbf{II}\mathbf{Y} = \mathbf{Y}\langle 1/\boldsymbol{\mu}\rangle. \tag{87}$$

Eqs. (85) and (86) are important for determining the real invariant matrix **G** and its 4×4 submatrices **L**, **S** and **H**. Eq. (87) characterizes the eigenvectors and the reciprocal base vectors as the eigenvectors of the matrices **T II** and $\mathbf{T}^{-1}$ **II**, respectively.

In the literature on 2-D anisotropic elasticity, the matrix corresponding to $\mathbf{Z}^+$ is often split into an upper square matrix **B** and a lower square matrix **A**. Adopting this notation for the present theory of anisotropic laminates, we write

$$\mathbf{Z} = \begin{bmatrix} \mathbf{B} & \overline{\mathbf{B}} \\ \mathbf{A} & \overline{\mathbf{A}} \end{bmatrix}. \tag{88}$$

Eqs. (39) and (46) become, respectively

$$\mathbf{\Omega} = \mathbf{A}^{\mathrm{T}}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{A}, \quad \mathbf{A}^{\mathrm{T}}\overline{\mathbf{B}} + \mathbf{B}^{\mathrm{T}}\overline{\mathbf{A}} = \mathbf{0}, \tag{89a, b}$$

while Eqs. (74) and (75) yield

$$\begin{bmatrix} 2\,\mathrm{Re}[\mathbf{B}\mathbf{\Omega}^{-1}\mathbf{B}^{\mathrm{T}}] & 2\,\mathrm{Re}[\mathbf{B}\mathbf{\Omega}^{-1}\mathbf{A}^{\mathrm{T}}] \\ 2\,\mathrm{Re}[\mathbf{A}\mathbf{\Omega}^{-1}\mathbf{B}^{\mathrm{T}}] & 2\,\mathrm{Re}[\mathbf{A}\mathbf{\Omega}^{-1}\mathbf{A}^{\mathrm{T}}] \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{4\times4} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{0}_{4\times4} \end{bmatrix},$$

$$\begin{bmatrix} 2\,\mathrm{Im}[\mathbf{B}\mathbf{\Omega}^{-1}\mathbf{B}^{\mathrm{T}}] & 2\,\mathrm{Im}[\mathbf{B}\mathbf{\Omega}^{-1}\mathbf{A}^{\mathrm{T}}] \\ 2\,\mathrm{Im}[\mathbf{A}\mathbf{\Omega}^{-1}\mathbf{B}^{\mathrm{T}}] & 2\,\mathrm{Im}[\mathbf{A}\mathbf{\Omega}^{-1}\mathbf{A}^{\mathrm{T}}] \end{bmatrix} = \begin{bmatrix} -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \\ \mathbf{S} & \mathbf{H} \end{bmatrix}.$$

Hence,

$$\mathbf{L} = 2\mathrm{i}\mathbf{B}\mathbf{\Omega}^{-1}\mathbf{B}^{\mathrm{T}}, \quad \mathbf{H} = -2\mathrm{i}\mathbf{A}\mathbf{\Omega}^{-1}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{S} = -\mathrm{i}(2\mathbf{A}\mathbf{\Omega}^{-1}\mathbf{B}^{\mathrm{T}} - \mathbf{I}). \tag{90}$$

Furthermore, Eq. (69a) yields

$$(\mathbf{S}^{\mathrm{T}} + \mathrm{i}\mathbf{I})\mathbf{B} = \mathbf{L}\mathbf{A}, \quad (\mathbf{S} + \mathrm{i}\mathbf{I})\mathbf{A} = -\mathbf{H}\mathbf{B}. \tag{91a, b}$$

Let **b** and **a** denote the four-vectors formed by the first and the last four elements of a zeroth-order eigenvector ξ. Then one may obtain from Eq. (17) the following expressions of **a** and **b** in terms of each other:

$$\mathbf{a} = \mathbf{Y}(\mu)\mathbf{C}^* \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \mathbf{b},$$

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} (\mathbf{C}^*)^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & \mu & 1 \end{bmatrix} \mathbf{a}. \tag{92a, b}$$

For a higher-order eigenvector $\xi^{[j]}$ ($j \geqslant 1$), the corresponding relations between $\mathbf{b}^{[j]}$ and $\mathbf{a}^{[j]}$ are determined by the derivative rule, i.e., by differentiating Eq. (92a,b) j-times with respect to μ. Notice that these relations depend on μ, and the form of the relations depends on the order j, whereas Eqs. (91a,b) and their complex conjugate relations have the same form for all eigenvectors in $\mathbf{Z}^+$ or $\overline{\mathbf{Z}}^+$.

Eqs. (92a,b) and the derived relations for the higher-order eigenvectors $\boldsymbol{\xi}^{[j]}$ imply that the matrices **B** and **A** uniquely determine each other. Let $\mathbf{A} = \boldsymbol{\Theta}\mathbf{B}$. Substitution into Eq. (89a) yields $\mathbf{B}^{\mathrm{T}}(\boldsymbol{\Theta} + \boldsymbol{\Theta}^{\mathrm{T}})\mathbf{B} = \boldsymbol{\Omega}$. Since $\boldsymbol{\Omega}$ is nonsingular, so must be **B**, and the invertibility of **A** then follows because **A** uniquely determines **B**. Finally, Eq. (90) implies that **L**, **H**, and $\mathbf{S} \pm \mathrm{i}\mathbf{I}$ are all nonsingular. Eqs. (91a,b) may be rewritten as

$$\mathbf{A} = -(\mathbf{S} - \mathrm{i}\mathbf{I})\mathbf{L}^{-1}\mathbf{B}, \quad \mathbf{B} = (\mathbf{S}^{\mathrm{T}} - \mathrm{i}\mathbf{I})\mathbf{H}^{-1}\mathbf{A}. \tag{93}$$

The identities (89b), (90)–(91) and (93), are valid for all anisotropic laminates, irrespective of degeneracy.

6. General solutions and intrinsic tensors of the eleven types of laminates

According to Eq. (67b), the intrinsic matrix **G** and its submatrices **L**, **H** and **S** are, respectively, the sums of the corresponding matrices $\mathbf{G}_k$, $\mathbf{L}_k$, $\mathbf{H}_k$ and $\mathbf{S}_k$ of the various eigenspaces. The latter can be obtained for all eight types of normal, abnormal and superabnormal eigenvalues according to Eq. (58). The results are found and shown in this section. We will frequently use the following identity, which is easily verified

$$\mathbf{W}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}\mathbf{W} = W\mathbf{W} - \delta(\mathbf{I} - \boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}). \tag{94}$$

Hence, for any scalar θ,

$$\theta\mathbf{J}\mathbf{W}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}\mathbf{J}^{\mathrm{T}} = \theta W\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}} - \delta\theta(\mathbf{J}\mathbf{J}^{\mathrm{T}} - \mathbf{J}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}\mathbf{J}^{\mathrm{T}}).$$

Differentiating the last equation N times with respect to μ and evaluating the result at an eigenvalue of multiplicity p, one obtains, for $N \leqslant p$,

$$(\mathrm{d}^N/\mathrm{d}\mu^N)(\theta\mathbf{J}\mathbf{W}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}\mathbf{J}^{\mathrm{T}}) = (\mathrm{d}^N/\mathrm{d}\mu^N)(\theta W\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}}), \tag{95a}$$

because $\delta = \delta' = \cdots = \delta^{(p-1)} = 0$, and

$$(\mathrm{d}^p/\mathrm{d}\mu^p)(\theta\mathbf{J}\mathbf{W}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}\mathbf{J}^{\mathrm{T}}) = (\mathrm{d}^p/\mathrm{d}\mu^p)(\theta W\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}}) - \delta^{(p)}\theta(\mathbf{J}\mathbf{J}^{\mathrm{T}} - \mathbf{J}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}\mathbf{J}^{\mathrm{T}}). \tag{95b}$$

Consider first the eigenspaces associated with normal eigenvalues of multiplicity $1 \leqslant p \leqslant 4$. As mentioned in the last section, if one chooses the eigenvectors to be $(\mathbf{J}\mathbf{w}/\sqrt{W})^{(N)}\boldsymbol{\rho}$, $N = 0, 1, \ldots, p-1$, then one obtains Eq. (80) instead of (52). Hence the expressions (A.1)–(A.4) for the pseudo-metrics and their inverses are simplified, and they depend only on δ and its nonvanishing derivatives, not on W and its derivatives. Observing that $(\boldsymbol{\omega}_k^{-1})_{rs} = 0$ whenever $r + s > p + 1$, Eqs. (58), (80) and (95a,b) yield

$$\begin{aligned}\mathbf{F}_k + \mathrm{i}\mathbf{G}_k &= \sum_{1\leqslant r\leqslant p}\ \sum_{1\leqslant s\leqslant 1+p-r} (\boldsymbol{\omega}_k^{-1})_{rs}(\mathbf{J}\mathbf{W}/\sqrt{W})^{(r-1)}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}(\mathbf{W}\mathbf{J}^{\mathrm{T}}/\sqrt{W})^{(s-1)} \\ &= \sum_{1\leqslant r\leqslant p} (\boldsymbol{\omega}_k^{-1})_{1r} \sum_{1\leqslant s\leqslant 1+p-r} (r-1, s-1)(\mathbf{J}\mathbf{W}/\sqrt{W})^{(r-t)}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}(\mathbf{W}\mathbf{J}^{\mathrm{T}}/\sqrt{W})^{(s-1)} \\ &= \sum_{1\leqslant r\leqslant p} (\boldsymbol{\omega}_k^{-1})_{1r}(\mathbf{J}\mathbf{W}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}\mathbf{W}\mathbf{J}^{\mathrm{T}}/W)^{(r-1)} = \sum_{1\leqslant s\leqslant p} (\boldsymbol{\omega}_k^{-1})_{1s}(\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}})^{(s-1)}.\end{aligned} \tag{96}$$

Using the new expressions of the inverses of the pseudo-metrics which do not depend on W, one obtains from Eq. (96) the matrix $\mathbf{F} + \mathrm{i}\mathbf{G}$ for normal eigenvalues of multiplicity from one to four. The results are given as Eqs. (B.1)–(B.4) in Appendix B.

For an abnormal double eigenvalue, Eq. (54b) yields

$$(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A2]} = (2/\delta'')\mathbf{J}\mathbf{W}'\mathbf{J}^{\mathrm{T}}. \tag{97}$$

For an abnormal eigenvalue of multiplicity 3 or 4, one stays with the eigenvectors as given by Eqs. (55a–d). Using $\mathbf{W}=\mathbf{0}$ and $W=0$, and substituting Eqs. (A.7b) and (A.8b) into (96), one obtains $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A3]}$ and $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A4]}$. They are given as Eqs. (B.6) and (B.7) of Appendix B.

For a superabnormal eigenvalue, Eq. (56b) yields

$$\begin{aligned}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[SA]} &= (4/\delta'''')(-\mathbf{J}\mathbf{W}'''\mathbf{J}^{\mathrm{T}} + 3\mathbf{J}'\mathbf{W}''\mathbf{J}^{\mathrm{T}} + 3\mathbf{J}\mathbf{W}''\mathbf{J}'^{T}) \\ &= (4/\delta'''')(\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}})''' + (12/5)(1/\delta'''')'(\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}})''.\end{aligned} \tag{98}$$

The final expressions of Eqs. (97), (B.6), (B.7), (98) for abnormal and superabnormal eigenvalues are found to be identical to those for a normal eigenvalue of the same multiplicity, i.e., $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N2]}$, $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N3]}$ and $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N4]}$ reduce to $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A2]}$, $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A3]}$ and $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A4]}$, respectively, when $\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}}$ vanishes for an abnormal eigenvalue, and $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A4]}$ reduces to $(\mathbf{F}+\mathrm{i}\mathbf{G})_{[SA]}$ when $(\mathbf{J}\mathbf{W}\mathbf{J}^{\mathrm{T}})'$ also vanishes for a superabnormal eigenvalue. These relations are also found a posteriori, that is, without a deductive proof.

As shown in Sections 3 and 4, the solution space of every anisotropic laminate is decomposable into orthogonal eigenspaces associated with simple or multiple eigenvalues which belong to one of the eight distinct types. Combinations of the various types of eigenvalues and the corresponding eigenspaces yield eleven mutually exclusive types of anisotropic laminates, each having a distinct analytical expression of the general solution and a distinct form of the pseudo-metric $\mathbf{\Omega} = [\![\mathbf{Z}^+, \mathbf{Z}^+]\!]$.

The eleven types of laminates are designated by notations starting with ND, D, ED and UD (nondegenerate, degenerate, extra-degenerate and ultra-degenerate) and followed by a sequence of symbols, one for each distinct eigenvalue of the laminate, denoting the multiplicity and, if not normal, abnormality or superabnormality. In the preceding analysis, only the eigensolutions associated with eigenvalues that have *positive* imaginary parts were explicitly described. These eigensolutions must be combined with the conjugate eigensolutions associated with the conjugate eigenvalues, in such a way that the respective coefficient functions f_k and g_k are related by $g_k(x+\bar{\mu}y) = \overline{f_k(x+\mu y)}$. Then the combined solutions always yield real values of the components of $\boldsymbol{\chi}$, $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ in Eqs. (5) and (6a,b). Each type of laminate has four complex conjugate pairs of eigensolutions with the orders varying from zero to three. Each eigensolution, regardless of the order, contains one independent arbitrary analytic function.

For all eleven types of laminates, the matrix $\mathbf{Z}^+$, the general solution vector $\boldsymbol{\chi}$, the pseudo-metric $\mathbf{\Omega}$ and the intrinsic tensor (1/2)i$\mathbf{G}$ are listed in Appendix C. These representations of the general solutions are fundamental to the analysis and solution of anisotropic laminate problems. Homogenous isotropic plates belong to the class of laminates that have superabnormal eigenvalues ±i. Such laminates are extra-degenerate (i.e., belong to the class [ED-4A] in the following notation). For these laminates, the present general solution reduces to Goursat's representation of biharmonic functions. Laminates with isotropic in-plane responses and anisotropic bending and twisting responses are usually called "quasi-isotropic." Such laminates generally have abnormal double eigenvalues ±i, and two distinct complex conjugate pairs of simple eigenvalues. These laminates are nondegenerate, and they belong to the class [ND1-1-2A]. Clearly, the general solution of anisotropic laminates, which manifests coupling between in-plane extension/shearing and out-of-plane bending/twisting, is far richer in content and variety than the lower dimension problem of 2-D anisotropic elasticity. The systematic and powerful analytical tools that have been developed for the plane elasticity problems may be modified and applied to anisotropic laminates, to produce a body of solutions no less copious and significant.

7. Isomorphisms and image laminates

In view of the similarity between the fourth-order differential equation governing the stress function of plane anisotropic elasticity and the corresponding equation governing the deflection function of anisotropic laminates, Lekhnitskii (1968, p. 283) observed that the bending problem of symmetric anisotropic laminates is closely related to the plane-stress problem of anisotropic elasticity. However, the exact nature of the relationship remains to be clarified. In this section, the relationship is examined in the broader context of laminates with generally nonvanishing coupling matrix B^*. It is shown that there is an isomorphism which associates every anisotropic laminate with an image laminate having closely related elasticity matrices, and which maps every equilibrium solution of the original laminate into a complementary solution of the image laminate. This transformation interchanges the bending variables with the in-plane variables. It also reverses the roles of the kinematical and kinetic variables. That is, the out-of-plane kinematical variables of the original solution corresponding to the in-plane kinetic variables of the image solution, and vice versa. Therefore, the boundary conditions of the original laminate and of the image laminate must be complementary, and cannot be identical.

We first recall that the subspace of solutions defined by the four eigenvectors in $\mathbf{Z}^+$ are identical in structure to the conjugate subspace defined by the elements of $\overline{\mathbf{Z}}^+$. Real-valued physical solutions are always obtained by combining mutually conjugate solutions of the two subspaces in a symmetric way. Besides the automorphism connecting the two subspaces associated with $\mathbf{Z}^+$ and $\overline{\mathbf{Z}}^+$, there is another one-to-one correspondence between the equilibrium solutions of an anisotropic laminate with the elasticity matrices A^*, B^*, D^*, and the equilibrium solutions of another laminate with the elasticity matrices D^*/h^2, $-(\mathsf{B}^*)^{\mathrm{T}}$ and $h^2\mathsf{A}^*$ (where h is a characteristic thickness parameter and we will henceforth take $h = 1$). In this correspondence relation, the roles of ∇F and ∇w are interchanged, as are the roles between the displacements u, v and the moment potentials Ψ_1 and Ψ_2. As a result, kinematical, kinetic and mixed boundary conditions also change into complementary conditions.

The formal similarity of the first three and last three elements of $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$

$$\boldsymbol{\phi} \equiv \{w_{,yy}, w_{,xx}, -w_{,xy}, F_{,yy}, F_{,xx}, -F_{,xy}\}^{\mathrm{T}},$$

$$\boldsymbol{\theta} \equiv \{\Psi_{1,x}, \Psi_{2,y}, \Psi_{1,y} + \Psi_{2,x}, -u_{,x}, -v_{,y}, -(u_{,y} + v_{,x})\}^{\mathrm{T}},$$

suggests the consideration of the following transformation

$$\boldsymbol{\xi}^{[j]} \to \langle \boldsymbol{\Upsilon}_4, -\boldsymbol{\Upsilon}_4\rangle \boldsymbol{\xi}^{[j]}, \quad \boldsymbol{\chi}^{[j]} \to \langle \boldsymbol{\Upsilon}_4, -\boldsymbol{\Upsilon}_4\rangle \boldsymbol{\chi}^{[j]}, \tag{99a, b}$$

$$\boldsymbol{\phi}^{[j]} \to \boldsymbol{\Upsilon}_6 \boldsymbol{\phi}^{[j]}, \quad \boldsymbol{\theta}^{[j]} \to -\boldsymbol{\Upsilon}_6 \boldsymbol{\theta}^{[j]}, \tag{99c, d}$$

$$\mathsf{C}^* \equiv \begin{bmatrix} \mathsf{D}^* & \mathsf{B}^* \\ \mathsf{B}^{*\mathrm{T}} & -\mathsf{A}^* \end{bmatrix} \to -\boldsymbol{\Upsilon}_6 \mathsf{C}^* \boldsymbol{\Upsilon}_6 \equiv \begin{bmatrix} \mathsf{A}^* & -\mathsf{B}^{*\mathrm{T}} \\ -\mathsf{B}^* & -\mathsf{D}^* \end{bmatrix}, \tag{99e}$$

$$\boldsymbol{\eta}^{[j]} \to \boldsymbol{\Upsilon}_2 \boldsymbol{\eta}^{[j]}, \quad \mathbf{M}(\mu) \to -\boldsymbol{\Upsilon}_2 \mathbf{M} \boldsymbol{\Upsilon}_2, \tag{99f, g}$$

$$\mathbf{E}(\mu) \to -\boldsymbol{\Upsilon}_6 \mathbf{E}(\mu) \langle \boldsymbol{\Upsilon}_4, -\boldsymbol{\Upsilon}_4\rangle, \quad \boldsymbol{\Phi}(\mu) \to \boldsymbol{\Upsilon}_6 \boldsymbol{\Phi}(\mu) \langle \boldsymbol{\Upsilon}_4, -\boldsymbol{\Upsilon}_4\rangle, \quad \mathbf{P}(\mu) \to \boldsymbol{\Upsilon}_6 \mathbf{P}(\mu) \boldsymbol{\Upsilon}_2, \tag{99h, i, j}$$

where

$$\boldsymbol{\Upsilon}_2 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\Upsilon}_4 \equiv \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0}_{2\times 2} \end{bmatrix} \quad \boldsymbol{\Upsilon}_6 \equiv \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0}_{3\times 3} \end{bmatrix}$$

satisfy

$$\boldsymbol{\Upsilon}_2\boldsymbol{\Upsilon}_2 = \mathbf{I}_2, \quad \boldsymbol{\Upsilon}_4\boldsymbol{\Upsilon}_4 = \mathbf{I}_4, \quad \boldsymbol{\Upsilon}_6\boldsymbol{\Upsilon}_6 = \mathbf{I}_6, \quad \langle\boldsymbol{\Upsilon}_4, -\boldsymbol{\Upsilon}_4\rangle\langle\boldsymbol{\Upsilon}_4, -\boldsymbol{\Upsilon}_4\rangle = \mathbf{I}_8.$$

It is easily verified that Eqs. (7), (21)–(23) and (31a,b) remain satisfied by the transformed quantities. Furthermore, since $\mathrm{Det}[-\boldsymbol{\Upsilon}_2\ \mathbf{M}\ \boldsymbol{\Upsilon}_2] = -\mathrm{Det}[\mathbf{M}]$, the eigenvalues are unchanged under the transformation of Eq. (99). Consider a laminate with the elasticity matrices $\mathbf{A}^*$, $\mathbf{B}^*$ and $\mathbf{D}^*$, and an image laminate whose corresponding elasticity matrices are numerically equal to $\mathbf{D}^*$, $-\mathbf{B}^{*\mathrm{T}}$ and $\mathbf{A}^*$, respectively (discrepancies in physical dimensions may be taken care of by introducing appropriate dimensional multiplicative factors). Then the two laminates have the same eigenvalues, and their eigenvectors are related by Eq. (99a). For every eigensolution of the original laminate, there is an eigensolution of the image laminate given by Eqs. (99b–d). It follows that any linear combination of the eigensolutions, i.e., *any* equilibrium solution of the original laminate, determines a transformed solution of the image laminate according to Eq. (99). The boundary conditions of the two solutions also transform according to Eqs. (99b–d), i.e., the in-plane kinematical and kinetic variables are changed, respectively, into the kinetic and kinematical variables associated with the out-of-plane deformation. To find the boundary data u and v of the image laminate, one needs to integrate the boundary forces and moments of the original laminate to obtain the data of Ψ_1 and Ψ_2 as functions of the boundary curve length s. This may be done as follows. Let

$$\theta \equiv (1/2)(\Psi_{1,y} - \Psi_{2,x})$$

and let s denote the arc length along the boundary. Then it is easily shown that

$$\mathrm{d}/\mathrm{d}s(\Psi_2\mathbf{i} - \Psi_1\mathbf{j}) = M_n\mathbf{n} + (M_{ns} + \theta)\mathbf{s}, \quad \mathrm{d}\theta/\mathrm{d}s = Q_n, \tag{100a, b}$$

where $\mathbf{s}$ and $\mathbf{n}$ are the unit tangent and normal vectors along the boundary, M_n and M_{ns} are the normal and twisting moments per unit curve length, and Q_n is the resultant of τ_{nz} per unit curve length. Hence $\Psi_1(s)$ and $\Psi_2(s)$ may be obtained by integrating the data of $\mathrm{d}M_{ns}/\mathrm{d}s + Q_n$ and M_n along the boundary:

$$M_{ns} + \theta = \int (\mathrm{d}M_{ns}/\mathrm{d}s + Q_n)\,\mathrm{d}s, \quad \Psi_2\mathbf{i} - \Psi_1\mathbf{j} = \int \{M_n\mathbf{n} + (M_{ns} + \theta)\mathbf{s}\}\,\mathrm{d}s. \tag{101a, b}$$

Notice that $\mathrm{d}M_{ns}/\mathrm{d}s + Q_n$ and M_n are precisely the kinetic boundary data commonly required in the classical plate theory. Eq. (101) implies that they are equivalent to the boundary data of the moment potentials, which arise as naturally in the present formulation as the boundary data of in-plane displacements. Notice also that the relationship of Eq. (99e) is expressed in terms of $\mathbf{A}^*$, $\mathbf{B}^*$ and $\mathbf{D}^*$. In the conventional laminate theory based on the stiffness matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$, the isomorphism of the image laminates is not easily discerned.

Laminates with $\mathbf{B} = \mathbf{0}$, and therefore $\mathbf{B}^* = \mathbf{0}$, show no coupling of in-plane and out-of-plane responses. They form an important class which includes all homogeneous plates as well as all laminated plates that are symmetric with respect to the mid-plane. For such laminates, Eqs. (8), (9) and (12) yield $M_{12}(\mu) \equiv 0$. The function space of general solutions is decomposable into two orthogonal subspaces, each of dimension four, one associated with the bending–twisting stiffness matrix $\mathbf{D}^* = \boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Lambda}$, and the other with the in-plane compliance matrix $\mathbf{A}^* = \mathbf{A}^{-1}$. Eq. (23) is separated into the following two sets of *scalar* equations, each determining the eigensolutions belonging to one of the two orthogonal subspaces:

$$M_{11}(\mu)\xi_2^{[0]} = 0, \quad M_{11}(\mu)\xi_2^{[1]} + M'_{11}(\mu)\xi_2^{[0]} = 0, \tag{102a}$$

$$M_{22}(\mu)\xi_4^{[0]} = 0, \quad M_{22}(\mu)\xi_4^{[1]} + M'_{22}(\mu)\xi_4^{[0]} = 0. \tag{102b}$$

The two sets of equations characterize, respectively, the bending–twisting problem and the stretching problem, which are now uncoupled. These two uncoupled problems, determined respectively by $M_{11}(\mu) = \{-\mu^2, -1, \mu\}\mathbf{D}^*\{\mu^2, -1, \mu\}^{\mathrm{T}}$ and $M_{22}(\mu) = \{-\mu^2, -1, \mu\}(-\mathbf{A}^*)\{-\mu^2, -1, \mu\}^{\mathrm{T}}$, have essentially an identical mathematical structure except for the reversal of the kinematical and kinetic variables.

The isomorphism has important implications. It implies that all analytic solutions of 2-D elasticity with the compliance matrix $\mathbf{A}^*$, including those that can be found in the standard references of Muskhelishvili and Lekhnitskii, are transformed by Eq. (99) into deflection solutions of an image laminate having the flexural stiffness $\mathbf{D}^*$ numerically equal to $\mathbf{A}^*$, and subjected to boundary conditions that are complementary to the original problem. For example, the plane-stress solution of an infinite plate subjected to the given traction vector $\{t_x, t_y\} = \{f_1(s), f_2(s)\}$ along the boundary of an arbitrarily-shaped hole is transformed by Eq. (99) into the bending/twisting solution of the image plate with the deflection data $\{w_{,y}, -w_{,x}\} = \int \{f_1, f_2\}\,\mathrm{d}s$ along the same hole boundary. Therefore, the class of bending solutions of symmetric anisotropic laminates is coextensive with the class of plane-stress solutions of anisotropic elasticity. Through the replacement of the variables $\{-\epsilon_x, -\epsilon_y, -2\epsilon_{xy}, \sigma_x, \sigma_y, \sigma_{xy}\}$ of the latter problem by the variables $\{M_y, M_x, -2M_{xy}, \kappa_y, \kappa_x, -\kappa_{xy}\}$, and the compliance matrix $\mathbf{A}^*$ by the new bending stiffness $\mathbf{D}^*$, the plane-stress boundary-value problem of elasticity is changed exactly into the bending problem of plates. Although Lekhnitskii (1968) pointed out the striking similarity between the differential equations governing the unknown functions F and w in the two problems, the exact correspondence of the variables and the boundary conditions was not clarified. Otherwise there would not have been the need of a bending theory of classical plates separate from the 2-D theory of anisotropic elasticity.

Appendix A. Pseudo-metrics of the eigenspaces of the various types of eigenvalues

(I) Eigenspaces associated with normal eigenvalues—multiplicity one to four

$$\boldsymbol{\omega}_{[N1]} = [\delta' W], \quad \boldsymbol{\omega}_{[N1]}^{-1} = [1/(\delta' W)], \tag{A.1a, b}$$

$$\boldsymbol{\omega}_{[N2]} \equiv \begin{bmatrix} 0 & \delta'' W/2 \\ \delta'' W/2 & \delta''' W/6 + \delta'' W'/2 \end{bmatrix}, \tag{A.2a}$$

$$\boldsymbol{\omega}_{[N2]}^{-1} = (1/W) \begin{bmatrix} (2/3)(1/\delta'')' - 2W'/(\delta'' W) & 2/\delta'' \\ 2/\delta'' & 0 \end{bmatrix}, \tag{A.2b}$$

$$\boldsymbol{\omega}_{[N3]} \equiv \begin{bmatrix} 0 & 0 & \delta''' W/3 \\ 0 & \delta''' W/6 & \delta^{(4)} W/12 + \delta''' W'/3 \\ \delta''' W/3 & \delta^{(4)} W/12 + \delta''' W'/3 & \delta^{(5)} W/30 + \delta^{(4)} W'/6 + \delta''' W''/3 \end{bmatrix}, \tag{A.3a}$$

$$\boldsymbol{\omega}_{[N3]}^{-1} \equiv (3/W\delta''') \begin{bmatrix} -W''/W + 2(W'/W)^2 + (\delta''''/\delta''')^2/8 - \delta^{(5)}/(10\delta''') + W'\delta''''/(2W\delta''') & -2W'/W - \delta''''/(2\delta''') & 1 \\ -2W'/W - \delta''''/(2\delta''') & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{A.3b}$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_{[N4]} = [\![\{\boldsymbol{\xi}^{[0]}, \boldsymbol{\xi}^{[1]}, \boldsymbol{\xi}^{[2]}, \boldsymbol{\xi}^{[3]}\}, \{\boldsymbol{\xi}^{[0]}, \boldsymbol{\xi}^{[1]}, \boldsymbol{\xi}^{[2]}, \boldsymbol{\xi}^{[3]}\}]\!] = [\Omega_{ij}], \tag{A.4a}$$

where all elements Ω_{ij} vanish except the following

$$\Omega_{14} = \Omega_{41} = 3\Omega_{23} = 3\Omega_{32} = (1/4)\delta^{(4)} W,$$

$$\Omega_{24} = \Omega_{42} = (3/2)\Omega_{33} = (1/20)\{\delta^{(5)} W + 5\delta^{(4)} W'\},$$

$$\Omega_{34} = \Omega_{43} = (1/60)\{\delta^{(6)} W + 6\delta^{(5)} W' + 15\delta^{(4)} W''\},$$

$$\Omega_{44} = (1/140)\{\delta^{(7)} W + 7\delta^{(6)} W' + 21\delta^{(5)} W'' + 35\delta^{(4)} W'''\}.$$

Inversion of the matrix $\boldsymbol{\omega}_{[N4]}$ may be facilitated by using the following expression of δ, which is valid only for a quadruple root of $\delta = 0$:

$$\delta = -(A^*_{11}D^*_{11} - B^{*2}_{11})(\mu - \mu_0)^4(\mu - \bar{\mu}_0)^4.$$

Taking the derivative of δ up to the seventh order, evaluating the result at $\mu = \mu_0$, substituting into Eq. (A.4a), and then inverting the matrix, one obtains $\boldsymbol{\omega}^{-1}_{[N4]}$ whose components $\boldsymbol{\Omega}^{-1}_{ij}$ are given by

$$\begin{aligned}
\boldsymbol{\Omega}^{-1}_{11} &= (12/W\delta'''')\{2W'W''/W^2 - W'''/(3W) - 2(W'/W)^3 + 4(W''/W - 2W'^2/W^2)(\mu_0 - \bar{\mu}_0)^{-1} \\
&\quad - 20(W'/W)(\mu_0 - \bar{\mu}_0)^{-2} - 40(\mu_0 - \bar{\mu}_0)^{-3}\}, \\
\boldsymbol{\Omega}^{-1}_{12} &= \boldsymbol{\Omega}^{-1}_{21} = \{2(W'/W)^2 - W''/W + 8(W'/W)(\mu_0 - \bar{\mu}_0)^{-1} + 20(\mu_0 - \bar{\mu}_0)^{-2}\}, \\
\boldsymbol{\Omega}^{-1}_{22} &= 2\boldsymbol{\Omega}^{-1}_{13} = 2\boldsymbol{\Omega}^{-1}_{31} = -24W'/(W^2\delta'''') - 96(\mu_0 - \bar{\mu}_0)^{-2}/(W\delta''''), \\
\boldsymbol{\Omega}^{-1}_{23} &= \boldsymbol{\Omega}^{-1}_{32} = 3\boldsymbol{\Omega}^{-1}_{14} = 3\boldsymbol{\Omega}^{-1}_{41} = 12/(W\delta''''), \\
\boldsymbol{\Omega}^{-1}_{24} &= \boldsymbol{\Omega}^{-1}_{33} = \boldsymbol{\Omega}^{-1}_{42} = \boldsymbol{\Omega}^{-1}_{34} = \boldsymbol{\Omega}^{-1}_{43} = \boldsymbol{\Omega}^{-1}_{44} = 0.
\end{aligned} \tag{A.4b}$$

(II) Eigenspaces associated with abnormal eigenvalues—multiplicity two to four

$$\boldsymbol{\omega}_{[A2]} = [\![\mathbf{J}(\mu), \mathbf{J}(\mu)]\!] = \mathbf{M}'(\mu), \quad \boldsymbol{\omega}^{-1}_{[A2]} = (2/\delta'')\mathbf{W}'(\mu), \tag{A.5a, b}$$

$$\boldsymbol{\omega}_{[A3]} = \begin{bmatrix} W' & 0 & 0 \\ 0 & 0 & W'\delta'''/3 \\ 0 & W'\delta'''/3 & W'\delta''''/6 + W''\delta'''/3 \end{bmatrix}, \tag{A.6a}$$

$$\boldsymbol{\omega}^{-1}_{[A3]} = 3/(W'\delta''') \begin{bmatrix} \delta'''/3 & 0 & 0 \\ 0 & -(1/2)\delta''''/\delta''' - W''/W' & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{A.6b}$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_{[A4]} = \begin{bmatrix} W' & 0 & 0 & 0 \\ 0 & 0 & 0 & W'\delta''''/4 \\ 0 & 0 & W'\delta''''/6 & W''\delta''''/4 + W'\delta^{(5)}/10 \\ 0 & W'\delta''''/4 & W''\delta''''/4 + W'\delta^{(5)}/10 & W'''\delta''''/4 + 3W''\delta^{(5)}/20 + W'\delta^{(6)}/20 \end{bmatrix}, \tag{A.7a}$$

$$\boldsymbol{\omega}^{-1}_{[A4]} = \begin{bmatrix} 1/W' & 0 & 0 & 0 \\ 0 & \boldsymbol{\Omega}^{-1}_{22} & \boldsymbol{\Omega}^{-1}_{23} & 4/(W'\delta'''') \\ 0 & \boldsymbol{\Omega}^{-1}_{23} & 6/(W'\delta'''') & 0 \\ 0 & 4/(W'\delta'''') & 0 & 0 \end{bmatrix}, \tag{A.7b}$$

where

$$\boldsymbol{\Omega}^{-1}_{22} = 6\{2W'\delta^{(5)} + 5W''\delta''''\}^2/(W'\delta'''')^3 - 4\{W'\delta^{(6)} + 3W''\delta^{(5)} + 5W'''\delta''''\}/(W'\delta'''')^2,$$

$$\boldsymbol{\Omega}^{-1}_{23} = -(6/5)\{2W'\delta^{(5)} + 5W''\delta''''\}/(W'\delta'''')^2.$$

(III) Eigenspace associated with a superabnormal eigenvalue

$$\boldsymbol{\omega}_{[SA]} = [\![\{\mathbf{J}, \mathbf{J}'\}, \{\mathbf{J}, \mathbf{J}'\}]\!] = \begin{bmatrix} \mathbf{0}_{2\times 2} & (1/2)\mathbf{M}''(\mu_0) \\ (1/2)\mathbf{M}''(\mu_0) & (1/6)\mathbf{M}'''(\mu_0) \end{bmatrix}, \tag{A.8a}$$

$$\boldsymbol{\omega}_{[SA]}^{-1} = (12/\delta'''')\begin{bmatrix} -\mathbf{W}'''/3 & \mathbf{W}'' \\ \mathbf{W}'' & \mathbf{0}_{2\times 2} \end{bmatrix}. \tag{A.8b}$$

Appendix B. Intrinsic tensor $F + iG$ associated with the various types of eigenvalues

$$(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N1]} = (1/\delta')\mathbf{JWJ}^{\mathrm{T}}, \tag{B.1}$$

$$(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N2]} = (2/\delta'')(\mathbf{JWJ}^{\mathrm{T}})' + (2/3)(1/\delta'')'\mathbf{JWJ}^{\mathrm{T}}, \tag{B.2}$$

$$(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N3]} = (3/\delta''')(\mathbf{JWJ}^{\mathrm{T}})'' + (3/2)(1/\delta''')'(\mathbf{JWJ}^{\mathrm{T}})' + 3\{(\delta'''')^2/(2\delta''')^3 - \delta^{(5)}/10(\delta''')^2\}\mathbf{JWJ}^{\mathrm{T}}, \tag{B.3}$$

$$\begin{aligned}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N4]} &= (4/\delta'''')(\mathbf{JWJ}^{\mathrm{T}})''' + (12/5)(1/\delta'''')'(\mathbf{JWJ}^{\mathrm{T}})'' + (12/31)(1/\delta'''')''(\mathbf{JWJ}^{\mathrm{T}})' \\ &\quad + (4/227)(1/\delta'''')'''\mathbf{JWJ}^{\mathrm{T}},\end{aligned} \tag{B.4}$$

$$(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A2]} = (2/\delta'')\mathbf{JW}'\mathbf{J}^{\mathrm{T}}, \tag{B.5}$$

$$\begin{aligned}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A3]} &= (1/W')\mathbf{J}(\mathbf{I}-\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}) - (3/2W')\{\delta''''/2(\delta''')^2 + W''/W'\}(W\mathbf{JWJ}^{\mathrm{T}})'' \\ &\quad + 1/(W'\delta''')\{W\mathbf{JWJ}^{\mathrm{T}} - \delta\mathbf{J}(\mathbf{I}-\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}})\mathbf{J}^{\mathrm{T}}\}''' \\ &= (3/\delta''')(\mathbf{JWJ}^{\mathrm{T}})'' + (3/2)(1/\delta''')'(\mathbf{JWJ}^{\mathrm{T}})',\end{aligned} \tag{B.6}$$

$$\begin{aligned}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A4]} &= (4/\delta'''')(\mathbf{JWJ}^{\mathrm{T}})''' + (12/5)(1/\delta'''')'(\mathbf{JWJ}^{\mathrm{T}})'' \\ &\quad + \{(24/25)(\delta^{(5)})^2/(\delta^{(4)})^3 - (4/5)\delta^{(6)}/(\delta^{(4)})^2\}(\mathbf{JWJ}^{\mathrm{T}})' \\ &= (4/\delta'''')(\mathbf{JWJ}^{\mathrm{T}})''' + (12/5)(1/\delta'''')'(\mathbf{JWJ}^{\mathrm{T}})'' + (12/31)(1/\delta'''')''(\mathbf{JWJ}^{\mathrm{T}})',\end{aligned} \tag{B.7}$$

$$\begin{aligned}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[SA]} &= (4/\delta'''')(-\mathbf{JW}'''\mathbf{J}^{\mathrm{T}} + 3\mathbf{J}'\mathbf{W}''\mathbf{J}^{\mathrm{T}} + 3\mathbf{JW}''\mathbf{J}'^{T}) \\ &= (4/\delta'''')(\mathbf{JWJ}^{\mathrm{T}})''' + (12/5)(1/\delta'''')'(\mathbf{JWJ}^{\mathrm{T}})''.\end{aligned} \tag{B.8}$$

Appendix C. Eigenvectors, general solutions and intrinsic tensors of eleven types of anisotropic laminates

The matrices $\boldsymbol{\omega}$ and $\mathbf{F}+\mathrm{i}\mathbf{G}$ that occur in the following expressions are given, respectively, in Appendices A and B for every type of eigenvalue.

(A) Non-degenerate laminates (four independent zeroth-order eigensolutions)

[ND-1-1-1-1]—Four distinct eigenvalues μ_i.

$$\mathbf{Z}^{+} = \{\mathbf{JW}\boldsymbol{\rho}(\mu_1), \mathbf{JW}\boldsymbol{\rho}(\mu_2), \mathbf{JW}\boldsymbol{\rho}(\mu_3), \mathbf{JW}\boldsymbol{\rho}(\mu_4)\}, \tag{C.1a}$$

$$\boldsymbol{\chi} = \mathrm{Re}\left[\sum_{1\leqslant k\leqslant 4} f_k(x+\mu_k y)\mathbf{JW}\boldsymbol{\rho}(\mu_k)\right], \tag{C.1b}$$

$$\boldsymbol{\Omega} = \langle \boldsymbol{\omega}_{[N1]}(\mu_1), \boldsymbol{\omega}_{[N1]}(\mu_2), \boldsymbol{\omega}_{[N1]}(\mu_3), \boldsymbol{\omega}_{[N1]}(\mu_4)\rangle, \tag{C.1c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}\sum_{1\leqslant k\leqslant 4}\mathbf{J}(\mu_k)\mathbf{W}(\mu_k)\mathbf{J}(\mu_k)^{\mathrm{T}}/\delta'(\mu_k). \tag{C.1d}$$

[ND-1-1-2A]—Two simple eigenvalues μ_1 and μ_2 and one double abnormal eigenvalue μ_3.

$$\mathbf{Z}^+ = \{\mathbf{JW\rho}(\mu_1), \mathbf{JW\rho}(\mu_2), \mathbf{J}(\mu_3)\}, \tag{C.2a}$$

$$\boldsymbol{\chi} = \mathrm{Re}\left[\sum_{1\leqslant k\leqslant 2} f_k(x+\mu_k y)\mathbf{JW\rho}(\mu_k) + \mathbf{J}(\mu_3)\{f_3(x+\mu_3 y), f_4(x+\mu_3 y)\}^{\mathrm{T}}\right], \tag{C.2b}$$

$$\mathbf{\Omega} = \langle \boldsymbol{\omega}_{[N1]}(\mu_1), \boldsymbol{\omega}_{[N1]}(\mu_2), \mathbf{M}'(\mu_3)\rangle, \tag{C.2c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N1]}(\mu_1) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N1]}(\mu_2) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A2]}(\mu_3). \tag{C.2d}$$

Notice that f_3 and f_4 in Eq. (C.2b) are independent functions of the same argument.

[ND-2A-2A]—Two double abnormal eigenvalues μ_1 and μ_2.

$$\mathbf{Z}^+ = \{\mathbf{J}(\mu_1), \mathbf{J}(\mu_2)\}, \tag{C.3a}$$

$$\boldsymbol{\chi} = \mathrm{Re}\left[\mathbf{J}(\mu_1)\{f_1(x+\mu_1 y), f_2(x+\mu_1 y)\}^{\mathrm{T}} + \mathbf{J}(\mu_2)\{f_3(x+\mu_2 y), f_4(x+\mu_2 y)\}^{\mathrm{T}}\right], \tag{C.3b}$$

$$\mathbf{\Omega} = \langle \mathbf{M}'(\mu_1), \mathbf{M}'(\mu_2)\rangle, \tag{C.3c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A2]}(\mu_1) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A2]}(\mu_2). \tag{C.3d}$$

(B) Degenerate laminates (require one first-order eigensolution)

[D-1-1-2]—Two simple eigenvalues μ_1 and μ_2 and one double normal eigenvalue μ_3.

$$\mathbf{Z}^+ = \left\{\mathbf{JW\rho}(\mu_1), \mathbf{JW\rho}(\mu_2), \mathbf{JW\rho}(\mu_3), (\mathbf{JW\rho})'(\mu_3)\right\}, \tag{C.4a}$$

$$\boldsymbol{\chi} = \mathrm{Re}\left[\sum_{1\leqslant k\leqslant 3} f_k(x+\mu_k y)\mathbf{JW\rho}(\mu_k) + \{f_4(x+\mu y)\mathbf{JW\rho}\}'(\mu_3)\right], \tag{C.4b}$$

$$\mathbf{\Omega} = \langle \boldsymbol{\omega}_{[N1]}(\mu_1), \boldsymbol{\omega}_{[N1]}(\mu_2), \boldsymbol{\omega}_{[N2]}(\mu_3)\rangle, \tag{C.4c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N1]}(\mu_1) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N1]}(\mu_2) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N2]}(\mu_3). \tag{C.4d}$$

[D-2A-2]—One double normal eigenvalue μ_1 and one double abnormal eigenvalue μ_2.

$$\mathbf{Z}_1 = \left\{\mathbf{JW\rho}(\mu_1), (\mathbf{JW\rho})'(\mu_1), \mathbf{J}(\mu_2)\right\}, \tag{C.5a}$$

$$\boldsymbol{\chi} = \mathrm{Re}\left[f_1(x+\mu_1 y)\mathbf{JW\rho}(\mu_1) + \{f_2(x+\mu y)\mathbf{JW\rho}\}'(\mu_1) + \mathbf{J}(\mu_2)\{f_3(x+\mu_2 y), f_4(x+\mu_2 y)\}^{\mathrm{T}}\right], \tag{C.5b}$$

$$\mathbf{\Omega} = \langle \boldsymbol{\omega}_{[N2]}(\mu_1), \mathbf{M}'(\mu_2)\rangle, \tag{C.5c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N2]}(\mu_1) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A2]}(\mu_2). \tag{C.5d}$$

[D-1-3A]—One simple eigenvalue μ_1 and one triple abnormal eigenvalue μ_2.

$$\mathbf{Z}^+ = \left\{\mathbf{JW\rho}(\mu_1), \boldsymbol{\xi}^{[0]}(\mu_2), (\mathbf{JW\hat{\rho}})'(\mu_2), (\mathbf{JW\hat{\rho}})''(\mu_2)\right\}, \tag{C.6a}$$

$$\boldsymbol{\chi} = \mathrm{Re}[f_1(x+\mu_1 y)\mathbf{JW}\boldsymbol{\rho}(\mu_1) + f_2(x+\mu_2 y)\boldsymbol{\xi}^{[0]}(\mu_2) + \{f_3(x+\mu y)\mathbf{JW}\hat{\boldsymbol{\rho}}\}'(\mu_2) + \{f_4(x+\mu y)\mathbf{JW}\hat{\boldsymbol{\rho}}\}''(\mu_2)], \tag{C.6b}$$

$$\boldsymbol{\Omega} = \langle \boldsymbol{\omega}_{[N1]}(\mu_1), \boldsymbol{\omega}_{[A3]}(\mu_2)\rangle, \tag{C.6c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N1]}(\mu_1) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A3]}(\mu_2). \tag{C.6d}$$

(C) Extra-degenerate laminates (require two higher-order eigensolutions)

[ED-1-3]—One simple eigenvalue μ_1 and one triple normal eigenvalue μ_2.

$$\mathbf{Z}^+ = \left\{\mathbf{JW}\boldsymbol{\rho}(\mu_1), \mathbf{JW}\boldsymbol{\rho}(\mu_2), (\mathbf{JW}\boldsymbol{\rho})'(\mu_2), (\mathbf{JW}\boldsymbol{\rho})''(\mu_2)\right\}, \tag{C.7a}$$

$$\boldsymbol{\chi} = \mathrm{Re}[f_1(x+\mu_1 y)\mathbf{JW}\boldsymbol{\rho}(\mu_1) + f_2(x+\mu_2 y)\mathbf{JW}\boldsymbol{\rho}(\mu_2) + \{f_3(x+\mu y)\mathbf{JW}\boldsymbol{\rho}\}'(\mu_2) + \{f_4(x+\mu y)\mathbf{JW}\boldsymbol{\rho}\}''(\mu_2)], \tag{C.7b}$$

$$\boldsymbol{\Omega} = \langle \boldsymbol{\omega}_{[N1]}(\mu_1), \boldsymbol{\omega}_{[N3]}(\mu_2)\rangle, \tag{C.7c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N1]}(\mu_1) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N3]}(\mu_2). \tag{C.7d}$$

[ED-2-2]—Two normal double eigenvalues μ_1 and μ_2.

$$\mathbf{Z}^+ = \left\{\mathbf{JW}\boldsymbol{\rho}(\mu_1), (\mathbf{JW}\boldsymbol{\rho})'(\mu_1), \mathbf{JW}\boldsymbol{\rho}(\mu_2), (\mathbf{JW}\boldsymbol{\rho})'(\mu_2)\right\}, \tag{C.8a}$$

$$\boldsymbol{\chi} = \mathrm{Re}[f_1(x+\mu_1 y)\mathbf{JW}\boldsymbol{\rho}(\mu_1) + \{f_2(x+\mu y)\mathbf{JW}\boldsymbol{\rho}\}'(\mu_1) + f_3(x+\mu_2 y)\mathbf{JW}\boldsymbol{\rho}(\mu_2) + \{f_4(x+\mu y)\mathbf{JW}\boldsymbol{\rho}\}'(\mu_2)], \tag{C.8b}$$

$$\boldsymbol{\Omega} = \langle \boldsymbol{\omega}_{[N2]}(\mu_1), \boldsymbol{\omega}_{[N2]}(\mu_2)\rangle, \tag{C.8c}$$

$$\mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N2]}(\mu_1) - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N2]}(\mu_2). \tag{C.8d}$$

[ED-4A]—One quadruple abnormal eigenvalue μ_0 which is not superabnormal.

$$\mathbf{Z}^+ = \{\boldsymbol{\xi}^{[0]}, \boldsymbol{\xi}^{[1]}, \boldsymbol{\xi}^{[2]}, \boldsymbol{\xi}^{[3]}\}, \quad \boldsymbol{\chi} = \mathrm{Re}[f_1\boldsymbol{\xi}^{[0]} + f_2\boldsymbol{\xi}^{[1]} + (f_3\mathbf{JW}\hat{\boldsymbol{\rho}})'' + (f_4\mathbf{JW}\hat{\boldsymbol{\rho}})'''], \tag{C.9a, b}$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_{[A4]}, \quad \mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[A4]}(\mu_0), \tag{C.9c, d}$$

where $\boldsymbol{\xi}^{[0]}, \boldsymbol{\xi}^{[1]}, \boldsymbol{\xi}^{[2]}$ and $\boldsymbol{\xi}^{[3]}$ are given by Eqs. (55a–d).

[ED-4AA]—One superabnormal eigenvalue μ_0.

$$\mathbf{Z}^+ = \{\mathbf{J}(\mu_0), \mathbf{J}'(\mu_0)\}, \quad \boldsymbol{\chi} = \mathrm{Re}[\mathbf{J}(\mu_0)\{f_1, f_2\}^{\mathrm{T}} + \mathbf{J}'(\mu_0)\{f_3, f_4\}^{\mathrm{T}}], \tag{C.10a, b}$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_{[SA]}, \quad \mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[SA]}. \tag{C.10c, d}$$

(D) Ultra-degenerate laminates (require three generalized eigensolutions)

[UD-4]—One normal quadruple eigenvalue.

$$\mathbf{Z}^+ = \{\mathbf{JW}\boldsymbol{\rho}, (\mathbf{J}'\mathbf{W}+\mathbf{JW}')\boldsymbol{\rho}, (\mathbf{J}''\mathbf{W}+2\mathbf{J}'\mathbf{W}'+\mathbf{JW}'')\boldsymbol{\rho}, (\mathbf{J}'''\mathbf{W}+3\mathbf{J}''\mathbf{W}'+3J'\mathbf{W}''+\mathbf{JW}''')\boldsymbol{\rho}\}, \tag{C.11a}$$

$$\boldsymbol{\chi} = \mathrm{Re}[f_1\mathbf{JW}\boldsymbol{\rho} + (f_2\mathbf{JW}\boldsymbol{\rho})' + (f_3\mathbf{JW}\boldsymbol{\rho})'' + (f_4\mathbf{JW}\boldsymbol{\rho})'''], \tag{C.11b}$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_{[N4]}, \quad \mathbf{G} = \mathrm{i}\mathbf{II} - 2\mathrm{i}(\mathbf{F}+\mathrm{i}\mathbf{G})_{[N4]}. \tag{C.11c, d}$$

The following equation gives a concise way to express the general solutions of various types of laminates in terms of the eigenvectors

$$\chi = \mathrm{Re}[\mathbf{Z}^{+}\mathbf{D}\mathbf{f}], \tag{C.12}$$

where **f** denotes the four-dimensional column vector formed by four arbitrary analytic functions of $x + \mu y$ and **D** is the identity matrix if the laminate is nondegenerate. For degenerate laminates, one has

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{d}/\mathrm{d}\mu \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{C.13}$$

The **D**-operator of extra-degenerate laminates is given by Eq. (C.14a) below, except when the set $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ contains two double normal eigenvalues (Type 2A) or one superabnormal eigenvalue (Type 4AA). In those exceptional cases **D** is given by Eq. (C.14b):

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \mathrm{d}/\mathrm{d}\mu & \mathrm{d}^2/\mathrm{d}\mu^2 \\ 0 & 0 & 1 & 2\mathrm{d}/\mathrm{d}\mu \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & \mathrm{d}/\mathrm{d}\mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{d}/\mathrm{d}\mu \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{C.14a, b}$$

Finally, for ultra-degenerate laminates one has

$$\mathbf{D} = \begin{bmatrix} 1 & \mathrm{d}/\mathrm{d}\mu & \mathrm{d}^2/\mathrm{d}\mu^2 & \mathrm{d}^3/\mathrm{d}\mu^3 \\ 0 & 1 & 2\mathrm{d}/\mathrm{d}\mu & 3\mathrm{d}^2/\mathrm{d}\mu^2 \\ 0 & 0 & 1 & 3\mathrm{d}/\mathrm{d}\mu \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{C.15}$$

After performing differentiation, the parameter μ in each analytic function $f_k(x + \mu y)$ and its derivatives must be replaced by the particular (simple or multiple) eigenvalue associated with the kth eigenvector in $\mathbf{Z}^{+}$.

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